An efficient syntax-preserving slide-based algorithm for program slicing

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Abstract

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Software systems continuously evolve over the years to avoid becoming less useful. However, evolution causes the code to diverge from the original design. Thus, the internal structure may change and reduce design quality. Refactoring is one of the major approaches that assist evolving software, yet keeping high design quality. The Extract Method refactoring enables to move a code fragment that can be grouped together into a separated method, replacing the old code with a call to the new method. For that, Program Slicing can be used.

In this thesis we present a backward, static, syntax-preserving slicing algorithm that elaborates on the Static single assignment form (SSA)-based slicing algorithm described in the PhD thesis of Ettinger. The slicing algorithm for program P involves 3 steps: converting P into an SSA form; computing its flow-insensitive slice; converting it back from SSA. The algorithm requires time polynomial in the size of the program. We defined a new slicing algorithm that is asymptotically faster and easier to implement than the SSA-based algorithm.
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Chapter 1

Introduction

Software maintenance is the most expensive activity in the software lifecycle. This process includes the following activities: adding new features and deletion of obsolete code (perfective), correcting errors (corrective), adapting to new environments (adaptive), and improving code quality (preventive) [11]. The latter increases software maintainability in order to prevent future problems. Software systems continuously evolve over the years to avoid becoming less useful [15]. However, software evolution causes the code to diverge from the original design, since code is often modified without referring to the effect of those modifications on the design. Hence, code eventually loses its structure and it becomes difficult to add new features or to modify the code without introducing new bugs [15]. Thus, even when starting with high quality of design (e.g., low coupling and high cohesion) the internal structure may change and reduce the design quality. Refactoring is one of the major approaches that assist evolving software, yet keeping high design quality [10].

Refactoring is defined as a process for restructuring a current software system to carry out improvements without changing its behavior. The main purpose in refactoring is to improve the quality of a program. This is achieved by reorganizing classes, variables, and methods across the class hierarchy to enable future adaptations and extensions, so that the source code can have better structure, readability, and understandability [18]. In addition to improving the internal structure of the code, refactoring can provide other benefits such as removing duplication of code, improving the design, making the code easier to understand and helping to program faster. There are many refactoring techniques such as composing methods (i.e., extract method, inline method), moving features between objects (i.e., move method), and simplifying method calls (i.e., rename method). The Extract Method refactoring enables to move a contiguous code fragment into a separated method, replacing the old code with a call to the new method. To extract a non-contiguous code fragment to a separate method one can use Program Slicing, a technique for simplifying programs by focusing on selected aspects of semantics [13].

1.1 Goal of this thesis

There are many slicing algorithms; see, for example, [19]. Most slicing algorithms use program-dependence graph (PDG) as its program representation. In this thesis we use and formalize a different program representation called slide-dependence graph (slideDG), which was introduced by Cozocaru [4]. Cozocaru showed a significant 10%-63% run-time improvement in using slideDG rather then PDG for many algorithms such as slicing, sliding, tucking, bucketing, etc.

In this thesis we present a backward, static, syntax-preserving slicing algorithm, which uses slideDG as its program representation. Our work elaborates on the static single assignment form (SSA)-based slicing algorithm described in the PhD thesis
of Ettinger [7]. The algorithm is proved to be semantics-preserving, yet it requires
time polynomial in the size of the program. We define a new slicing algorithm that
is asymptotically faster and easier to implement than the SSA-based algorithm.

1.2 Background

In what follows, we provide some needed background on slicing.

1.2.1 Program slicing

Program slicing was invented by Mark Weiser [20] as a method for producing a
subprogram that preserves a subset of the behavior of the original program. This
sub-program is called a slice, with respect to a certain slicing criterion. A slicing crite-
ron is a pair \(<p, V>\), where \(p\) is a certain point of interest in the program and \(V\) is a
set of variables. For all the examples in this chapter, the point of interest is marked
as a statement number in a given program. The slice consists of all the statements
in the program (up to the specified point of interest) that directly or indirectly effect
the value of the variables given in the slicing criterion. Any slice of a given pro-
gram preserves the following property: for the initial values of each variable in the
program, if the original program terminates, the slice also terminates with the same
final values at point \(p\) of each variable in the program. In this thesis, the point of
interest \(p\) is always the end of a code-fragment of a given program which we choose
to slice from, therefore we only use a set of variables as the slicing criterion.

The leading program representation for calculating a slice is PDG, program de-
pendence graph. The PDG represents a program as a graph in which the nodes
are statements and predicate expressions and the edges incident to a node represent
either the data values on which the node’s operations depend and the control con-
ditions on which the execution of the operations depends [9]. In this thesis, we use
a variation on the PDG called SlideDG, which will be described later. Note that the
original intention of slicing was debugging [20, 21], but since then there have been
many other applications to slicing such as software maintenance, testing, program
differencing, refactoring [2].

Our first slicing example\(^1\) is given in Listings 1.1-1.3 where the variables even
and odd respectively count the number of even and odd values in a given array of
integers. Note that the statements from the original program that are not in the slice
are written as an empty statement (blank line).

\(^1\)In this thesis, the programs are written in a deterministic variation on Dijkstra’s guarded com-
mands [6], as defined in [7].
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1 i := 0;
2 even := 0;
3 odd := 0;
4 while i < a.length do
5 if a[i] % 2 == 0 then
6 even := even + 1
7 else
8 odd := odd + 1
9 fi;
10 i := i + 1
11 od

LISTING 1.1: Original program P1

1 i := 0;
2 even := 0;
3 while i < a.length do
4 if a[i] % 2 == 0 then
5 even := even + 1
6 else
7 odd := odd + 1
8 fi;
9 i := i + 1
10 od

LISTING 1.2: A slice of P1 with V={even}

LISTING 1.3: A slice of P1 with V={odd}

Static vs. dynamic slicing An important distinction is between static and dynamic slicing. Static slicing computes a slice with no assumptions on the program's input. Dynamic slicing computes a slice for a specific input [14]. In what follows we refer to the program in Listing 1.4.

1 read(n);
2 i := 1;
3 result := 1;
4 while i <= n do
5 result := result * i;
6 i := i + 1
7 od;
8 print(result)

LISTING 1.4: Original program P2

In static slicing, all irrelevant statements to the slicing criterion are not included in the slice, as shown in Listing 1.5 regarding the program in Listing 1.4.
1 read(n);
2 i := 1;
3 result := 1;
4 while i <= n do
5   result := result * i;
6   i := i + 1
7  od;
8 print(result)

LISTING 1.5: A static slice of <8, result> for P2

In dynamic slicing, as shown in Listing 1.6 regarding the program in Listing 1.4, only the relevant statements to the specific input are in the slice.

1 read(n);
2 i := 1;
3 result := 1;
4
5
6
7
8 print(result)

LISTING 1.6: A dynamic slice for n=0 of <8, result> for P2

**Backward vs. forward slicing** Another important distinction between two possible directions of slicing is backward and forward slicing. A backward slice of a program and a slicing criterion <p, V> includes all statements and predicates that may have affected the value of the variables in V up to the program point p. Backward slicing can be helpful for debugging. A forward slice of a program and a slicing criterion <p, V> contains all the statements and predicates that may be affected by the values of the variables in V from the program point p. It can be useful to determine which statements will be affected by changing the value of the variable in the slicing criterion, but it is usually not executable. Examples for both directions are given in Listings 1.7 and 1.8 (both referring to original program P2 in Listing 1.4).

1 read(n);
2 i := 1;
3
4 while i <= n do
5   i := i + 1
6  od
7
8 print(result)

LISTING 1.7: Backward slice of <8, i>

1
2
3 result := 1;
4
5 result := result * i;
6
7
8 print(result)

LISTING 1.8: Forward slice of <3, result>

**Syntax-preserving vs. amorphous slicing** A syntax-preserving slice is constructed by deleting all statements irrelevant to the slicing criterion from the original program, thus preserving the original program’s syntax. By using syntax-preserving
slicing we can produce a non-contiguous slice, which we call a *substatement* of the original program. In contrast, an amorphous (semantics-preserving) slice does not have to preserve the program’s syntax, and can produce a smaller slice by changing some of the program’s statements yet still preserving the original program’s behavior [12].

```plaintext
1 read (n);
2 if n >= 0 then
3   Skip
4 else
5   n := n * -1
6 fi;
7 print (n)
```

**LISTING 1.9:** Original program P3

```plaintext
1 read (n);
2 if n < 0 then
3   n := n * -1
4 fi;
5 print (n)
```

**LISTING 1.10:** Amorphous slice of <6, n> for P3

In this thesis we consider syntax-preserving, backward, static slicing algorithm.

### 1.2.2 Programming notations and representation

Our input program for the slicing algorithm is a compound statement. Each statement in our language is one the following:

- **Assignment** (LHS, RHS): Two sequences, one for the left-hand side of the assignment and another for the right-hand side of the assignment, used to support multiple assignment.
- **Sequential Composition** (S1, S2): Two statements executed one after the other.
- **If** (BoolExp, Then, Else): Boolean expression, a statement for the "then" case and a statement for the "else" case.
- **Do** (BoolExp, LoopBody): Boolean expression and a statement for the loop body.
- **Skip**: An empty statement.

Formal declarations for the statements in the language are given in Chapter 2.

### 1.2.3 Data flow analysis

Data flow analysis is a form of static program analysis and is a process of collecting run-time information about data in programs without executing them [22, 17]. The program representation with data-flow problems is usually the control flow graph (CFG) [3].

**Definition 1** (CFG). Control flow graph is a program representation (in the form of a directed graph) of all the possible program’s paths during its execution. Each node of the CFG represent a program statement with two additional nodes, entry and exit. Each edge of the CFG \((n_1, n_2)\), represent potential control flow from \(n_1\) to \(n_2\). Normal nodes have one successor, the exit node has no successors, and a predicate node corresponding to a conditional or a loop condition has two successors, where each edge is labeled *True* or *False*. The root of the CFG is the entry node, which is a predicate that has the exit node as its false successor.

An example for a CFG is shown in Figure 1.1.
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Definition 2 (Final-def node). Given a program S, a CFG node n and a variable v, n is considered a final-def node for v iff v is defined in n and there exists a path from n to the exit free of definitions of v.

Definition 3 (Final-use node). Given a program S, a CFG node n and a variable v, n is considered a final-use node for v iff v is used in n and each path from n to the exit is free of definitions of v.

We now describe two classical data-flow analyses: reaching definitions analysis and live-variable analysis.

Reaching definitions analysis

Reaching definitions analysis is a data-flow analysis of computing the set of definitions that reach a point in the program (where a definition is a pair of (variable, statement number)). Let v be a variable in the left-hand side of an assignment in a given program point p. This assignment may reach a program point p’ if v is defined at p, used at p’, and there is a CFG path from p to p’ free from definitions of v. The set of definitions that reach the exit of a statement S is calculated by: \( \text{RD}_{\text{out}}(S) = (\text{RD}_{\text{in}}(S) \setminus \text{Kill}(S)) \cup \text{Gen}(S) \), where \( \text{RD}_{\text{in}} \) is the set of definitions at the entry of S, \( \text{Kill} \) is the set of definitions removed by S, and \( \text{Gen} \) is the set of definitions generated by S. Reaching definitions information is usually given by the pair \( (\text{RD}_{\text{in}}, \text{RD}_{\text{out}}) \) which holds the reaching definitions at the entry of a program and at the end of a program, respectively. In Listing 1.4, \( \text{RD}_{\text{out}}(5) \) is calculated using:

- \( \text{RD}_{\text{in}}(5) = \{(n, 1), (i, 2), (\text{result}, 3), (\text{result}, 5), (i, 6)\} \)
- \( \text{Kill}(5) = \{(\text{result}, 3), (\text{result}, 5)\} \)
- \( \text{Gen}(5) = \{(\text{result}, 5)\} \)

Therefore: \( \text{RD}_{\text{out}}(5) = \{(n, 1), (i, 2), (\text{result}, 5), (i, 6)\} \), where each definition is given by the pair (variable, statement number).
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Live-variable Analysis

Live-variable analysis is another data-flow analysis computed in the direction opposite to the flow of control in a program (backward analysis) [1]. In live-variable analysis, a variable \( v \) is considered live at the exit of a program point \( p \) if there exists a CFG path (that does not define \( v \)) from \( p \) to a use of \( v \) [17]. Otherwise, \( v \) is considered dead at \( p \). The set of variables live at the entry of a statement \( S \) is calculated by: 

\[
LV_{\text{entry}}(S) = (LV_{\text{exit}}(S) \setminus \text{def}(S)) \cup \text{use}(S),
\]

where \( LV_{\text{exit}} \) is the set of variables live at the exit of \( S \), \( \text{def} \) is the set of variables defined in \( S \), and \( \text{use} \) is the set of variables whose values may be used in \( S \) prior to any definition of the variable. In Listing 1.4, \( LV_{\text{entry}}(S) \) is calculated using:

- \( LV_{\text{exit}}(5) = \{\text{result, i}\} \)
- \( \text{def}(5) = \{\text{result}\} \)
- \( \text{use}(5) = \{\text{result, i}\} \)

Thus: \( LV_{\text{entry}}(5) = \{\text{result, i}\} \).

In this thesis, we use both reaching definitions and live-variable analyses for the proof of our slicing algorithm.

1.2.4 Slips and slides

Our algorithm represents a statement by a set of slides. Before we show our algorithm, we present two terms that are used in the algorithm by means of an example slip and slide. Let \( S \) be the statement: “if \( \text{num} > 0 \) then \( \text{pos} := 1 \) else \( \text{pos} := 0 \) fi”. The slips of \( S \) are: “\( \text{pos} := 1 \)”, “\( \text{pos} := 0 \)”, and “if \( \text{num} > 0 \) then \( \text{pos} := 1 \) else \( \text{pos} := 0 \) fi”, while the slides of \( S \) are: “if \( \text{num} > 0 \) then \( \text{pos} := 1 \) fi” and “if \( \text{num} > 0 \) then Skip else \( \text{pos} := 0 \) fi”.

A slip is any part of a statement which is in itself a statement. A slide is an assignment with all the control-statements (if, do) and sequential-compositions in which it is contained. In case of a multiple-assignment, each assignment (with the relevant control-statements) is considered a slide. In reference to a program representation of a tree, a slip is a subtree and a slide is a path from the root to a certain leaf (described in Chapter 2). For further examples, let us refer back to P1 in Listing 1.1. Some slips there are: “\( \text{odd} := 0 \)”, “\( i := i + 1 \)”, and also “if \( a[i] \) % 2 == 0 then \( \text{even} := \text{even} + 1 \) else \( \text{odd} := \text{odd} + 1 \) fi”; and some slides are given in Listings 1.11-1.13.

```plaintext
1 Skip ;
2 Skip ;
3 odd := 0 ;
4 Skip
5
6
7
8
9
10
11
```

LISTING 1.11: Slide example 1
Chapter 1. Introduction

We also define the union of slides into a statement. Given a statement $S$ and two slides $S_m$ and $S_n$ of $S$, each program point in the resulting statement contains the statement that is not Skip from the corresponding programs points in $S_m$ and $S_n$. In the case that both corresponding programs points in $S_m$ and $S_n$ contains Skip, the resulting statement also contains Skip. An example is given in Listing 1.14 for the union of the slides from Listings 1.11 and 1.12.

Formal definitions of slip and slide are given in Chapter 2. In this thesis we use slides as a program representation for our algorithm, and both slips and slides in the proof of our algorithm.

1.2.5 Slide dependence

The algorithm we have defined is slide-based and uses a representation of a program called a slide-dependence graph (slideDG) which was introduced by Cozocaru [4]. For the following definitions we use two terms:

- For a slide $s$ and a variable $v$ we say that $v$ is defined in $s$ if $v$ is on the left-hand side of the assignment in $s$.

- For a slide $s$ and a variable $v$ we say that $v$ is used in $s$ if $v$ is on the right-hand side of the assignment in $s$, or is used in one of the control-statements of $s$.

Definition 4 (Slide Dependence). There is a slide dependence due to variable $v$ between two slides $S_m$ and $S_n$ of CFG nodes $m$ and $n$ respectively, iff there is a definition of $v$ in $m$ that reaches any node $n' \in S_n$ and $v$ is used in $n'$.
Definition 5 (SlideDG). A Slide-dependence graph (SlideDG) is a program representation (in the form of a directed graph) with slides as nodes, each slide $S_n$ corresponds to a CFG node $n$, and there is an edge from slide $S_n$ to slide $S_m$ iff $S_m$ is slide-dependent on $S_n$.

A formal definition of slide-dependence and slide-dependence graph is given in Chapter 2, and an example\textsuperscript{2} is given in Figure 1.2.

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (1) at (0,3) {1};
  \node (2) at (3,3) {2};
  \node (3) at (6,3) {3};
  \node (6) at (0,0) {6};
  \node (8) at (3,0) {8};
  \node (10) at (6,0) {10};

  \draw[->] (1) to (2);
  \draw[->] (2) to (3);
  \draw[->] (1) to (6);
  \draw[->] (6) to (8);
  \draw[->] (8) to (10);
  \draw[->] (6) to (8);
  \draw[->] (8) to (10);
\end{tikzpicture}
\caption{SlideDG of P1 (from Listing 1.1).}
\end{figure}

Definition 6 (Final-def Slide). A slide $S_n$ is called a final-def slide of a statement $S$ and a set of variables $V$ iff the assignment of $S_n$ is to a variable $v$ such that $v \in V$, performed at a CFG node $n$, and is reaching the exit of $S$.

In Figure 1.2 and for $V=$\{even\}, slides 2 and 6 are the final-def slides.

1.2.6 Dafny

The formal framework of this thesis is written in Dafny (1.9.9), a programming language that verifies that the programmer writes correct code with no run-time errors using a verifier. The verifier is used to verify the correctness of the program, while the programmer is writing the code [16]. The verification relies on program specification and annotations such as pre- and post-conditions, loop invariants, assertions, and lemmas.

1.2.7 Static single assignment

The static single assignment (SSA) form, introduced by Cytron et al., is a program representation in which every variable is assigned only once and defined before it is used [5]. Usually, a program is converted to SSA form in order to perform some sort of a program analysis, and then converted back to its original form. In Cytron’s approach there are two steps in order to convert a program $S$ and a set of variables $V$ into an SSA form: Inserting assignments called $\Phi$-functions to certain points in the program (control-flow merge points), where the operands to a $\Phi$-function indicate which assignments to a certain variable $v$ reach the merge point. Then, renaming each instance of $v \in V$ to a new name $v_i$ (and increasing $i$ for each instance). In our simple language, instead of using $\Phi$-functions (which are not executable), they can be separated into two assignments. For an IF statement, we use an assignment at the end of each branch, and for a DO statement we use an assignment before the loop

\textsuperscript{2}In this thesis, the slide-dependence graph does not include self edges, since their presence has no influence on the results of our algorithm.
and at the end of its body. Examples are given in Listings 1.15, 1.16, and 1.17. The algorithm to converting a program into an SSA form and back from SSA used in this thesis was presented by Ettinger [7] and will be demonstrated next.

```plaintext
1 read (x);
2 if x > 0 then
3   y := 1
4 else
5   y := 2
6 fi;
7 print (y)
```

LISTING 1.15: A simple program

```plaintext
1 read (x1);
2 if x1 > 0 then
3   y2 := 1
4 else
5   y3 := 2
6 fi;
7 y4 := Φ(y2,y3);
8 print (y4)
```

LISTING 1.16: SSA form

```plaintext
1 read (x1);
2 if x1 > 0 then
3   y2 := 1;
4   y4 := y2
5 else
6   y3 := 2;
7   y4 := y3
8 fi;
9 print (y4)
```

LISTING 1.17: SSA form used in this thesis

### 1.2.8 SSA-based slicing

An SSA-based slicing algorithm was presented and proved to be semantics-preserving by Ettinger [7]. The algorithm first converts a statement into SSA form. Then, it computes the flow-insensitive slice on a given set of variables (as explained next). Finally, it returns back from SSA resulting in a flow-sensitive slice. The algorithm has the complexity of $O(n^3)$, where $n$ is the number of statements in the program. We demonstrate the three steps of the algorithm using the example in Listing 1.18.

```plaintext
1 i := 0;
2 sum := 0;
3 prod := 1;
4 while i < a.length do
5   sum := sum + a[i];
6   prod := prod + a[i];
7   i := i + 1
8 od
```

LISTING 1.18: Original statement $S$

**Translate a statement into SSA form**

For the first step, given a statement $S$ and a set of variables $V$ the algorithm translates $S$ into SSA form, while keeping a mapping of each variable and its set of instances. In our example, the statement $S'$ in Listing 1.19 is the SSA form result of the statement $S$ in Listing 1.18.
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Compute the flow-insensitive slice

For the second step, given a statement $S$ and a set of variables $V$, the algorithm computes the smallest possible slide-independent superset $V'$ of $V$, and then computes the union of slides of $S$ on $V'$. In our example, the statement $SV'$ in Listing 1.20 is the flow-insensitive slice of the statement $S'$ in Listing 1.19 on $V'$, where $V'$ consists of the live-on-exit instance of each variable in $V$. Note that in SSA form each variable has at most one live instance in any point in the program.

Translate a statement back from SSA form

For the third and final step, given a statement $S$, the algorithm translates $S$ back from SSA form, while using the previous mapping of each variable to its set of instances. In our example, the statement $res$ in Listing 1.21 is the translated result of the statement $SV'$ in Listing 1.20.
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To and from SSA specification

In order to succeed in the proof of correctness of our algorithm, we formulated a formal functional specification for both the To-SSA and From-SSA algorithms, as follows:

RemoveEmptyAssignments(Rename($S'$, $XLs$, $X$, $globS$)) = $S$

where:

- $S$: The original statement.
- $S'$: The SSA version of $S$.
- $X$: A sequence containing all variables defined in $S$.
- $XLs$: A mapping between each variable from $X$ to its set of instances.
- $globS$: The set of glob($S$) (used to verify that the mapping is valid).
- Rename: A function that renames each instance in a given statement to its original variable. In the case where an assignment was renamed into a self-assignment, that assignment is replaced with an empty-assignment.
- RemoveEmptyAssignments: A function that removes all empty assignments in a given statement.

However, this specification is bounded by two preconditions:

1. No self assignments: If there is a self assignment in $S$, for example $x := x$, its SSA version in $S'$ can be $x1 := x2$. According to our specification, the Rename function renames the assignment back to $x := x$ and replaces the self assignment with an empty assignment. Then, the RemoveEmptyAssignments function removes the empty assignment and our specification would not hold.

2. No empty assignments: If there is an empty assignment in $S$, its SSA version in $S'$ would still be an empty assignment. According to our specification, the RemoveEmptyAssignments function removes the assignment and our specification would not hold.

For example, $S$ is the statement from Listing 1.18, $S'$ is the statement from Listing 1.19, $X$ is the sequence of variables [$i$, $sum$, $prod$], and $XLs$ is the mapping of variables [{i1, i4, i7}, {sum2, sum4, sum5}, {prod3, prod4, prod6}]. When we use Rename on $SV'$ from Listing 1.20, we get the statement in Listing 1.22. Then, when we use RemoveEmptyAssignments on the result of Rename on $SV'$ we get res from Listing 1.21.

```
1 i := 0;
2 sum := 0;
3 [] := [];  
5 while i < a.length do
6 sum := sum + a[i];
7  
8 i := i + 1;
9 [] := []
10 od
```

**LISTING 1.22: Rename($SV'$, $XLs$, $X$)**
Formal definitions of both Rename and RemoveEmptyAssignments are given in Appendix A.1.4.

1.3 Contributions

The main contributions of this thesis are as follows:

1. It formalizes the definitions of slide-dependence graph and varSlide-dependence graph, and the connection between the two.

2. It provides a functional specification for the transition of a program into SSA, the computation of its flow-insensitive slice, and the transition back from SSA.

3. It proves that any slicing algorithm that meets the requirements listed above is syntax-preserving, in particular the SSA-based algorithm presented in 1.2.8, thus compatible to code-motion refactoring.

4. It provides a new, efficient, semantics- and syntax-preserving slicing algorithm.

Thesis outline The rest of the thesis is structured as follows: Chapter 2 provides a formal framework for this thesis. Chapter 3 elaborates on the correspondence between the definitions in the formal framework. Chapter 4 presents our slide-based slicing algorithm. In Chapter 5 we provide the syntax-preservation proof of our algorithm. Chapter 6 concludes the thesis and suggests ideas for future work.
Chapter 2

Formal framework

In this chapter, we provide the formal framework for our thesis. Here we formally define the slide-dependence graph and varSlide-dependence graph of a program.

2.1 Running example

We illustrate the formal framework of this thesis and the correctness proof of our slicing algorithm (Chapter 5) using the example in Listing 2.1 that calculates the sum of numbers in a given non-empty array, finds its max, and checks whether the array is in a strictly ascending order. The example is an adaptation of an example presented in the VerifyThis competition\textsuperscript{1}.

```
1 max := a[0];
2 sum := a[0];
3 i := 1;
4 count := 0;
5 while i < a.length do
6   if a[i] > max then
7     max := a[i]
8   else
9     skip
10  fi;
11 sum := sum + a[i];
12 if max > a[i-1] then
13   count := count + 1
14 else
15   skip
16 fi;
17 i := i + 1
18 od;
19 if count + 1 == a.length then
20   isSorted := true
21 else
22   isSorted := false
23 fi
```

LISTING 2.1: Original program \textit{S}

\textsuperscript{1}https://www.pm.inf.ethz.ch/research/verifythis.html
Figure 2.1: SlideDG of S from Listing 2.1

\begin{verbatim}
1 max1 := a[0];
2 sum2 := a[0];
3 i3 := 1;
4 count4 := 0;
5 max5, sum5, i5, count5 := max1, sum2, i3, count4;
6 while i5 < a.length do
7    if a[i5] > max5 then
8       max7 := a[i5];
9       max6 := max7
10      else
11         max6 := max5
12        fi;
13    sum8 := sum5 + a[i5];
14    if max6 > a[i5-1] then
15       count10 := count5 + 1;
16       count9 := count10
17    else
18       count9 := count5
19      fi;
20    i11 := i5 + 1;
21    max5, sum5, i5, count5 := max6, sum8, i11, count9
22 od;
23 if count5 + 1 == a.length then
24    isSorted13 := true;
25    isSorted12 := isSorted13
26 else
27    isSorted14 := false;
28    isSorted12 := isSorted14
29 fi
\end{verbatim}

Listing 2.2: SSA Version of S from Listing 2.1
2.2 Programming notations and representation

As mentioned in the previous chapter, our input program is a compound statement. The declarations in Dafny for the statements in the language are as follows:

\[
\textbf{datatype} \quad \text{Statement} = \\
\quad \text{Assignment} (\text{LHS: seq<Variable>, RHS: seq<Expression>}) \\
\quad \mid \text{SeqComp} (S1: Statement, S2: Statement) \\
\quad \mid \text{IF} (B0: BooleanExpression, Sthen: Statement, Selse: Statement) \\
\quad \mid \text{DO} (B: BooleanExpression, Sloop: Statement) \\
\quad \mid \text{Skip} \\
\quad \mid \text{LocalDeclaration} (L: seq<Variable>, S0: Statement) \\
\quad \mid \text{Live} (L: seq<Variable>, S0: Statement) \\
\quad \mid \text{Assert} (B: BooleanExpression)
\]

where:

\[
\textbf{type} \quad \text{Variable} = \text{string} \\
\textbf{type} \quad \text{Expression} = (\text{State} \rightarrow \text{Value}, \text{set<Variable>, string}) \\
\textbf{type} \quad \text{BooleanExpression} = (\text{State} \rightarrow \text{bool}, \text{set<Variable>}) \\
\textbf{datatype} \quad \text{Value} = \text{Int}(i: \text{int}) \mid \text{Bool}(b: \text{bool}) \\
\textbf{type} \quad \text{State} = \text{map<Variable, Value>}
\]

The input program language for the SSA-based algorithm uses the entire definition of Statement, whereas in this thesis we only use Assignment, SeqComp, IF, DO, and Skip as our core language.

We represent a statement as a tree, where leaves represent assignments or skips and inner nodes represent composite statements. A label is a sequence of numbers representing a path from the root to a certain statement in the tree. In Dafny we define a label as a sequence of branches, where a Branch is the number 1 or 2. An example for the representation of a statement as a tree is shown in Figure 2.3. The declaration in Dafny is the following:
newtype Branch = b : int | 1 ≤ b ≤ 2

type Label = seq<Branch>

We can use the definition of label when defining a slip of a statement. The formal definition of slip in Dafny is described below. The slipOf function uses a number of predicates defined in Appendix A.1.1. Valid checks if S is a valid statement (for example, |LHS| = |RHS| if the statement is an assignment); Core checks if S is either Assignment, SeqComp, IF, DO, or Skip; and ValidLabel checks if l is a valid label in the statement S.

function slipOf (S: Statement, l: Label): Statement
  requires Valid(S) ∧ Core(S)
  requires ValidLabel(l, S)
  ensures Valid(slipOf(S, l)) ∧ Core(slipOf(S, l))
  decreases 1

  | if l = [] then S
  else
match $S$
  case SeqComp($S_1, S_2$) $\Rightarrow$
    if $l_1[0] = 1$ then slipOf($S_1, l_1[1..]$)
    else slipOf($S_2, l_1[1..]$)
  case IF($B_0, S_{\text{then}}, S_{\text{else}}$) $\Rightarrow$
    if $l_1[0] = 1$ then slipOf($S_{\text{then}}, l_1[1..]$)
    else slipOf($S_{\text{else}}, l_1[1..]$)
  case DO($B, S_1$) $\Rightarrow$
    slipOf($S_1, l_1[1..]$)

2.3 Program analysis

In this section we provide our formal definitions of reaching definitions and live-variable analysis presented in Chapter 1.

2.3.1 Reaching definitions

The formal definition of reaching definitions in Dafny is described below. We denote the set of definitions that reach the beginning of a statement as reaching definitions in, and the set of definitions that reach the end of a statement as reaching definitions out.

Reaching definitions in

function ReachingDefinitionsIn($S$: Statement, $l$: Label):
  set $\langle$(Variable, Label)$\rangle$
  requires Valid($S$) $\land$ Core($S$)
  requires ValidLabel($l$, $S$)
  {
    ReachingDefinitionsInRec($S$, $l$, $S$, $l$, $\{\}$, $\{\}$)
  }

function ReachingDefinitionsInRec(slipOf$S$: Statement,
  $l_1$: Label, $S$: Statement, $l_2$: Label,
  rdIn: set $\langle$(Variable, Label)$\rangle$): set $\langle$(Variable, Label)$\rangle$
  requires Valid($S$) $\land$ Core($S$)
  requires Valid(slipOf$S$) $\land$ Core(slipOf$S$)
  requires ValidLabel($l_1$, slipOf$S$)
  requires ValidLabel($l_2$, $S$)
  {
    if $l_1 = \{\}$ then
      if IsDO(slipOf$S$) then
        rdIn + ReachingDefinitionsOutRec($S$, $\{\}$, $l_2+[1]$)
      else rdIn
    else
      assert $\neg$IsAssignment(slipOf$S$) $\land$ $\neg$IsSkip(slipOf$S$);
    match slipOf$S$
      case SeqComp($S_1, S_2$) $\Rightarrow$
        if $l_1[0] = 1$ then ReachingDefinitionsInRec($S_1$, $l_1[1..]$,
          $S$, $l_2+[1]$, rdIn)
        else ReachingDefinitionsInRec($S_2$, $l_1[1..]$,
      case IF($B_0, S_{\text{then}}, S_{\text{else}}$) $\Rightarrow$
        if $l_1[0] = 1$ then ReachingDefinitionsInRec($S_{\text{then}}$, $l_1[1..]$,
          $S_{\text{then}}$, $l_1[1..]$,
function ReachingDefinitionsInRec(S: Statement, l: Label, rdIn)
    requires Valid(S) ∧ Core(S)
    | validLabel(l, S)
    | validLabel(rdIn)
| 
* rdIn := ReachingDefinitionsIn(S, l) |
* ReachingDefinitionsInRec(S, l, rdIn + ReachingDefinitionsOutRec(S, { }, l2+[1]))
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Chapter 2. Formal framework

function RDKill(V: seq<Variable>, rdIn: set<(Variable, Label)>):
    set<(Variable, Label>)
    |
    if V = [] then rdIn
    else
        var s := set p | p in rdIn ∧ V[0] = p.0;
        RDKill(V[1..], rdIn − s)
    |
function RDGen(V: seq<Variable>, l: Label):
    set<(Variable, Label)>
    |
    set v | v in V • (v, l)
|

For brevity, we denote the ReachingDefinitionsIn function by RD_IN(S, l), ReachingDefinitionsInFor function by RD_IN_FOR(S, l, v), ReachingDefinitionsOut function by RD_OUT(S, l), and ReachingDefinitionsOutFor function by RD_OUT_FOR(S, l, v).

For the following examples on program S in Listing 2.1, we denote the labels in the sets of RD_IN, RD_IN_FOR, RD_OUT and RD_OUT_FOR as the number of statement in the given program:

• RD_IN(S, [2,2,2,2]) = {(max, 1), (sum, 2), (i, 3), (count, 4)}.
• RD_IN_FOR(S, [2,2,2,2], count) = {(count, 4)}
• RD_OUT(S, []) = {(max, 1), (sum, 2), (i, 3), (count, 4), (max, 7), (sum, 11), (count, 13), (i, 17), (isSorted, 20), (isSorted, 22)}.
• RD_OUT_FOR(S, [], count) = {(count, 4), (count, 13)}.

2.3.2 Liveness

The formal definition of liveness analysis in Dafny is described below. We denote the set of variables that are live at the beginning of a statement as live on entry, and the set of variables that are live at the end of a statement as live on exit.

Live on entry

function LiveOnEntry(S: Statement, lvExit: set<Variable>):
    set<Variable>
    |
    requires Valid(S) ∧ Core(S)
    |
    match S |
    case Assignment(LHS,RHS) ⇒ LVKill(LHS, liveOnExit) + LVGen(RHS)
    case Skip ⇒ lvExit
    case SeqComp(S1,S2) ⇒ LiveOnEntry(S1, LiveOnEntry(S2, lvExit))
    case IF(B0,Sthen,Selse) ⇒ LiveOnEntry(Sthen, lvExit) +
    LiveOnEntry(Selse, lvExit) + B0.1
    case DO(B,Sloop) ⇒ lvExit + LiveOnEntry(Sloop, {}) + B.1
    |
    } |

Kill and Gen functions

function LVKill(V: seq<Variable>, lvExit: set<Variable>):
    set<Variable>
Chapter 2. Formal framework

\[
\begin{align*}
\text{if } V = [] & \text{ then } lvExit \\
\text{else} & \\
& \text{ var } s := \text{ set } v \mid v \text{ in } lvExit \land V[0] = v; \\
& \text{ LVKill}(V[1..], lvExit - s)
\end{align*}
\]

function \text{LVGen}(E: \text{ seq<Expression>}) : \text{ set<Variable>}
\{
GetRHSVariables(E)
\}

The function \text{GetRHSVariables} can be found in Appendix A.1.1.

Live on exit

function \text{LiveOnExit}(S: \text{ Statement}, \text{lvExit}: \text{ set<Variable>}, l: \text{ Label}) : \text{ set<Variable>}
\text{requires Valid}(S) \land \text{Core}(S)
\text{requires ValidLabel}(l, S)
\{
\text{if } l = [] \text{ then } lvExit\text{ else assert } \neg \text{IsAssignment}(S) \land \neg \text{IsSkip}(S); \\
\text{match } S \{
\text{case } \text{SeqComp}(S1, S2) \Rightarrow \\
\text{ if } l[0] = 1 \text{ then } \text{LiveOnExit}(S1, \text{LiveOnEntry}(S2, lvExit), l[1..]) \\
\text{ else } \text{LiveOnExit}(S2, lvExit, l[1..])
\text{case } \text{IF}(B0, S\text{then}, S\text{else}) \Rightarrow \\
\text{ if } l[0] = 1 \text{ then } \text{LiveOnExit}(S\text{then}, lvExit, l[1..]) \\
\text{ else } \text{LiveOnExit}(S\text{else}, lvExit, l[1..])
\text{case } \text{DO}(B, S\text{loop}) \Rightarrow \\
\text{LiveOnExit}(S\text{loop}, lvExit + \\
\text{LiveOnEntry}(S\text{loop}, lvExit), l[1..])
\}
\}

For brevity, we denote the \text{LiveOnEntry} function by \text{LV_ENTRY}(S, lvExit) and \text{LiveOnExit} function by \text{LV_EXIT}(S, lvExit, l).

For example:
- \text{LV_ENTRY}(S, \{\text{max5, sum5, i5, count5, isSorted12}\}) = \{}.
- \text{LV_EXIT}(S, \{\text{max5, sum5, i5, count5, isSorted12}\}, []) = \{\text{max5, sum5, i5, count5, isSorted12}\}.
- \text{LV_EXIT}(S, \{\text{max5, sum5, i5, count5, isSorted12}\}, [2,2,2,2,1]) = \{\text{max5, sum5, i5, count5\}.}

2.4 Slides

We recall the definition of a slide from Chapter 1 as an assignment with all the control-statements (if, do) and sequential-compositions in which it is contained. The formal definition of \text{Slide} is as follows, where Label is the label of the assignment
in the slide, and \textit{Variable} is the variable defined in the slide. For that we use two
accessors, \textit{SlideLabel} and \textit{SlideVariable}, which can be found in Appendix A.1.1.

\textbf{type} \textit{Slide} = (\textit{Label}, \textit{Variable})

\subsection*{2.4.1 Slide dependence graph}

A formal definition of slide dependence in Dafny is described below. The \textit{SlideDependence} predicate uses \textit{SlidesOf} which returns all the slides of a statement; \textit{def} which returns the set of variables defined in a specific statement; \textit{SlideLabels} which returns a set of the slide's label and all the labels of the control-statements (if, do) and sequential-compositions in which it is contained; \textit{UsedVars} which returns all the variables used for a specific label and statement; and \textit{ReachingDefinition} which checks if a certain pair of (variable, label1) is in \textit{RD_IN(S, label2)}. Full definitions can be found in Appendix A.1.2.

\textbf{predicate} \textit{SlideDependence} (\textit{Sm: Slide}, \textit{Sn: Slide}, \textit{S: Statement})
\begin{align*}
\text{requires } & \text{Valid(S)} \land \text{Core(S)} \\
\text{requires } & \text{Sm in SlidesOf(S, def(S))} \land \text{Sn in SlidesOf(S, def(S))} \\
\text{var } & v := \text{SlideVariable(Sm)}; \\
\exists & l \cdot l \text{ in SlideLabels(Sn, S)} \land v \text{ in UsedVars(S, l)} \land \text{ReachingDefinition(S, SlideLabel(Sm), l, v)} \\
\end{align*}

\begin{align*}
\text{requires } & \text{Valid(S)} \land \text{Core(S)} \\
\text{requires } & \text{ValidLabel(l1, S)} \land \text{ValidLabel(l2, S)} \\
\text{var } & (v, l1) \text{ in ReachingDefinitionsIn(S, l2)} \\
\end{align*}

For example, the variable \textit{max} is defined in slide 1, used in the label \textit{l} = [2,2,2,2,1,1,1,1] which is one of the labels of slide 7, and \textit{(max, [1])} is in \textit{RD_IN(S, l)}, therefore slide 7 is slide-dependent on slide 1.

The formal definition of a \textit{slide-dependence graph} is as follows, where \textit{Statement} is the statement represented by the graph; \textit{set<Slide>} is the set of slides of the graph (nodes); and \textit{map<Slide, set<Slide>>} is the mapping between each slide to its predecessors (edges). For that we use three accessors: \textit{SlideDGStatement}, \textit{SlideDGSlides} and \textit{SlideDGMap}, which can be found in Appendix A.1.2.

\textbf{type} \textit{SlideDG} = (\textit{Statement}, \textit{set<Slide>}, \textit{map<Slide, set<Slide>>>)

\textbf{function} \textit{SlideDGOf} (\textit{S: Statement}): \textit{SlideDG}
\begin{align*}
\text{requires } & \text{Valid(S)} \land \text{Core(S)} \\
\text{var } & \text{slides} := \text{SlidesOf(S, def(S))}; \\
\text{var } & \text{m} := \text{map s \mid s in slides \bullet} \\
& \text{SlideDependencePredecessorsOf(s, S)}; \\
\text{var } & (S, \text{slides}, \text{m}) \\
\end{align*}

\textbf{function} \textit{SlideDependencePredecessorsOf} (\textit{Sn: Slide}, \textit{S: Statement})
\begin{align*}
\text{requires } & \text{Valid(S)} \land \text{Core(S)} \\
\end{align*}
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2.4.2 Paths in a slide dependence graph

We describe a path in a slideDG between two slides as SlideDGPath with the following definition:

datatype SlideDGPath = Empty | Extend(SlideDGPath, Slide)

Then, we describe the reachability between two slides as SlideDGReachable with the following definition:

predicate SlideDGReachable(slideDG: SlideDG, from: Slide, to: Slide, slides: set<Slide>)

predicate SlideDGReachableVia(slideDG: SlideDG, from: Slide, via: SlideDGPath, to: Slide, slides: set<Slide>)

decreases via

function SlideDGPredecessors(slideDG: SlideDG, n: Slide): set<Slide>

Another important distinction we make is between finalDefSlides and the rest. Given a statement S and its slide-dependence graph slideDG, it returns the set of slides that are reaching to the exit of S.

function FinalDefSlides(S: Statement, V: set<Variable>): set<Slide>

function FinalDefSlidesOfVariable(S: Statement, v: Variable): set<Slide>
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\begin{verbatim}
var rdIn := set v | v in def(S) • (v, []);
var rdOutv := set pair | pair in ReachingDefinitionsOutRec(S, rdIn, []) ∧ pair.0 = v;
var slidesRdOutv := set pair | pair in rdOutv • (pair.1, pair.0);
set slide | slide in slidesRdOutv * SlideDGSlides(slideDG)
\}

For example, FinalDefSlides(S, {isSorted}) is the set of slides \{20, 22\}.

We derive the following definition from reaching definitions and final-def slides:

**Definition 7.** The set of slides \textit{RD\textsubscript{OUT}}(S, []) is the set of all the final-def slides of \textit{S}.

2.5 VarSlides

The program representation we use for the SSA form of a program is the \textit{VarSlide-Dependence graph}. Similar to a \textit{Slide-Dependence graph}, the \textit{VarSlide-Dependence graph} uses \textit{VarSlides} as nodes. The formal definition of \textit{VarSlide} is as follows, where \textit{Variable} is the variable defined in the \textit{varSlide}, and \textit{VarSlideTag} is the tag of the \textit{varSlide}, regular when \textit{Variable} is defined only once and phi when \textit{Variable} is defined twice (control-flow merge point). For that we use two accessors, \textit{VarSlideVariable} and \textit{VarSlideTag}, which can be found in Appendix A.1.3.

\texttt{datatype VarSlideTag = Phi | Regular
type VarSlide = (Variable, VarSlideTag)}

Note that a \textit{varSlide} does not contain a label, therefore if a \textit{varSlide} is regular it has only one valid label in a given statement, and if a \textit{varSlide} is phi it has two valid labels in the statement.

2.5.1 VarSlide dependence graph

The definitions of \textit{varSlide dependence} and \textit{varSlide-dependence graph} are as follows:

**Definition 8 (VarSlide Dependence).** There is a \textit{varSlide dependence} between two \textit{varSlides} \textit{S\textsubscript{m}} and \textit{S\textsubscript{n}} iff there is a variable \textit{v} defined in \textit{S\textsubscript{m}} and used in \textit{S\textsubscript{n}}.

**Definition 9 (VarSlideDG).** A \textit{VarSlide-dependence graph} (\textit{VarSlideDG}) is a program representation (in the form of a directed graph) with \textit{varSlides} as nodes and there is an edge between \textit{varSlides} \textit{S\textsubscript{n}} and \textit{S\textsubscript{m}} iff \textit{S\textsubscript{m}} is \textit{varSlide-dependent} on \textit{S\textsubscript{n}}.

A formal definition of \textit{varSlide dependence} in Dafny is described below. The \textit{VarSlideDependence} predicate uses \textit{VarSlideLabels} which returns a set of the \textit{varSlide’s} label and all the labels of the control-statements (if, do) and sequential-compositions in which it is contained. Full definition can be found in Appendix A.1.3.

\texttt{predicate VarSlideDependence(Sm: VarSlide, Sn: VarSlide, S: Statement)
    requires Valid(S) \land Core(S)
    {\n    var v := VarSlideVariable(Sm);
    \exists l • l in VarSlideLabels(Sn, S) \land v in UsedVars(S, l)
    }\n}

For example, the variable \textit{max5} is defined in the \textit{varSlide} of \textit{max5} and used in the label \textit{l} = \{2,2,2,2,1,2,1,1,1,1,1\} of the \textit{varSlide} of \textit{max7}, therefore the \textit{varSlide} of \textit{max7} is \textit{varSlide-dependent} on the \textit{varSlide} of \textit{max5}.
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The formal definition of varSlide-dependence graph is as follows, where Statement is the statement represented by the graph, set<VarSlide> is the set of varSlides of the graph (nodes), and map<VarSlide, set<VarSlide>> is the mapping between each varSlide to its predecessors (edges). For that we use three accessors, VarSlideDGStatement, VarSlideDGVarSlides and VarSlideDGMp, which can be found in Appendix A.1.3. The VarSlideDGO function uses VarSlidesOf which returns all the varSlides of a statement, and can also be found in Appendix A.1.3.

**type** VarSlideDG = (Statement, set<VarSlide>, map<VarSlide, set<VarSlide>>)  

**function** VarSlideDGO(T: Statement): VarSlideDG  
**requires** Valid(T) ∧ Core(T)  
{  
var varSlides := VarSlidesOf(T, def(T));  
var m := map s | s in varSlides • VarSlideDependencePredecessorsOf(s, T);  
(T, varSlides, m)  
}  

**function** VarSlideDependencePredecessorsOf(Sn: VarSlide, T: Statement): set<VarSlide>  
**requires** Valid(T) ∧ Core(T)  
**requires** Sn in VarSlidesOf(T, def(T))  
{  
set Sm | Sm in VarSlidesOf(T, def(T)) ∧ VarSlideDependence(Sm, Sn, T)  
}  

### 2.5.2 Paths in a varSlide dependence graph

We express the existence of a path in varSlideDG between two varSlides as VarSlideDGPath with the following definition:

**datatype** VarSlideDGPath = Empty | Extend (VarSlideDGPath, VarSlide)  

Then, we describe the reachability between two varSlides as VarSlideDGReachable or VarSlideDGReachablePhi with the following definitions:

**predicate** VarSlideDGReachable(varSlideDG: VarSlideDG, from: VarSlide, to: VarSlide, S: set<VarSlide>)  
{  
∃ via: VarSlideDGPath • VarSlideDGReachableVia(varSlideDG, from, via, to, S)  
}  

**predicate** VarSlideDGReachableVia(varSlideDG: VarSlideDG, from: VarSlide, via: VarSlideDGPath, to: VarSlide, S: set<VarSlide>)  
**decreases** via  
{  
match via  
  case Empty ⇒ from = to  
  case Extend(prefix, n) ⇒ n in S ∧ to in VarSlideDGNeighbours(varSlideDG, n) ∧ VarSlideDGReachableVia(varSlideDG, from, prefix, n, S)  
}
function VarSlideDGNeighbours(varSlideDG: VarSlideDG, n: VarSlide):
    set<VarSlide>
    requires n in VarSlideDGMap(varSlideDG)
    { VarSlideDGMap(varSlideDG)[n] }

predicate VarSlideDGReachablePhi(varSlideDG: VarSlideDG, from: VarSlide, to: VarSlide, S: set<VarSlide>)
    { \exists via : VarSlideDGPath • VarSlideDGReachableViaPhi(varSlideDG, from, via, to, S) }

predicate VarSlideDGReachableViaPhi(varSlideDG: VarSlideDG, from: VarSlide, via: VarSlideDGPath, to: VarSlide, S: set<VarSlide>)
    decreases via
    { match via case Empty ⇒ from = to case Extend(prefix, n) ⇒ n in S ∧ to in VarSlideDGNeighbours(varSlideDG, n) ∧ n.1 = Phi ∧ VarSlideDGReachableVia(varSlideDG, from, prefix, n, S) }

2.6 Correspondence between slideDG and varSlideDG

Given a statement S and its corresponding SSA version S’, we describe the connection between a slide in the slideDG of S and its corresponding varSlide in the varSlideDG of S’ using the following function:

    requires Valid(S) ∧ Valid(S’)
    requires Core(S) ∧ Core(S’)
    requires ValidXLSs(glob(S’), XLS, X)
    requires S = RemoveEmptyAssignments(Rename(S’, XLS, X, glob(S)))
    requires slide in SlideDGSlides(SlideDGOf(S))
    { var v, l := SlideVariable(slide), SlideLabel(slide);
      var l’ := VarLabelOf(S, S’, l, XLS, X);
      var v’ := InstanceOf(S’, l’, v, XLS, X);
      (v’, Regular) }

We first find the variable v and label l of the given slide. Then, we find l’s corresponding varLabel l’ (using the VarLabelOf function), and finally we find the instance i (of variable v) defined in l’ (using the InstanceOf function). Every slide in the slideDG of S has exactly one corresponding regular varSlide in the varSlideDG of S’ (and vice versa), therefore VarSlideOf is considered an invertible function. The declarations of functions used in VarSlideOf can be found in Appendix A.1.4.

For example, given slide 7:
• The label of slide 7 is \( l = [2,2,2,1,1,1,1] \).
• The variable of slide 7 is \( v = \text{max} \).
• The varLabel of \( l \) is \( l' = [2,2,2,1,2,1,1,1,1,1,1] \).
• The instance of \( v \) defined at \( l' \) is \( \text{max7} \).
• Therefore, the varSlide corresponding to slide 7 for \( \text{max} \) is \((\text{max7}, \text{Regular})\).

Further correspondences between the two graphs are described in the next chapter.
Chapter 3

Properties of slide dependence graphs

In this chapter we provide a deeper look at the correspondence between slideDG and varSlideDG of a program.

3.1 Reaching definitions and liveness

We present two theorems that describe the correspondence between reaching definitions and liveness analysis. Both theorems use the following definition (regular frontier):

\[
RF(v') = \begin{cases} 
\emptyset & \text{v' is not in def(S)} \\
\{\text{varSlide of v'}\} & \text{v' is a regular instance} \\
\text{regular predecessors of the varSlide of v'} & \text{v' is a phi instance}
\end{cases}
\]

Note that for a given varSlide \(v_{\text{Slide}}\), for each predecessor \(p\) of \(v_{\text{Slide}}\): if \(p\) is a regular varSlide, then it is in the regular predecessors of \(v_{\text{Slide}}\); if \(p\) is a phi varSlide, then the regular predecessors of \(p\) are in the regular predecessors of \(v_{\text{Slide}}\).

**Theorem 1** (Reaching definitions and liveness for exit). Let \(S\) be a statement, and \(S'\) be its SSA form. Let \(l', v'\) be a valid label in \(S'\) and a live on exit instance of \(S'\) at \(l'\), respectively. Let \(v, l\) be the variable of \(v'\) and the corresponding label of \(l'\) using the \(\text{VarLabelOf}\) function, respectively.

We then say that all varSlides of \(\text{RD\_OUT\_FOR}(S, l, v) = RF(v')\).

**Examples**

Let us demonstrate this theorem on Listings 2.1 and 2.2 using the following examples:

1. For the first example, let \(l', v'\) be 15 and \(\text{sum8}\), respectively, as \(\text{sum8}\) is a live on exit instance of \(S'\) at 15. Let \(v, l\) be \(\text{sum}\) and 13, respectively, as \(\text{sum}\) is the variable of \(\text{sum8}\) and 13 is the label of 15 in \(S\). The set \(\text{RD\_OUR\_FOR}(S, 13, \text{sum})\) is \(\{(\text{sum}, 11)\}\), and its corresponding set of varSlides is \(\{\text{sum8}\}\). The varSlide of \(\text{sum8}\) is Regular, therefore \(RF(\text{sum8})\) is \(\{\text{sum8}\}\).

2. For the second example, let \(l', v'\) be 13 and \(\text{max6}\), respectively, as \(\text{max6}\) is a live on exit instance of \(S'\) at 13. Let \(v, l\) be \(\text{max}\) and 11, respectively, as \(\text{max}\) is the variable of \(\text{max6}\) and 11 is the label of 13 in \(S\). The set \(\text{RD\_OUR\_FOR}(S, 11, \text{max})\)
is \{((max,1), (max,7))\}, and its corresponding set of varSlides is \{max1, max7\}. The varSlide of max6 is Phi, therefore RF(max6) is \{max1, max7\}.

3. For the third and final example, let \(l', v'\) be the empty label \([\]\) and isSorted12, respectively, as isSorted12 is a live on exit instance of \(S'\) at \([\]\), meaning its a live on exit instance of the entire statement \(S'\). Let \(v, l\) be isSorted and \([\]\), respectively, as isSorted is the variable of isSorted12 and \([\]\) is the label of \([\]\) in \(S\). The set RD_OUT_FOR(S, \([\]\), isSorted) is \{((isSorted,20), (isSorted,22))\} (the final def slides of isSorted in \(S\) according to Definition 7), and its corresponding set of varSlides is \{isSorted13, isSorted14\}. The varSlide of isSorted12 is Phi, therefore RF(isSorted12) is \{isSorted13, isSorted14\}.

The exact details of the proof are not immediately needed for understanding the theorem, therefore the formal proof can be found in Appendix A.2.1. However, here we provide a short description of the proof.

In the case that slipOf(S, \(l\)) is an assignment and \(v\) is defined in that assignment, the set RD_OUT_FOR(S, \(l\), \(v\)) includes only the slide of \((v, l)\). The label \(l'\) is the var-label of \(l\), therefore cannot be a label of a Phi assignment. Thus, the var-slide of \((v, l)\) is Regular, therefore RF(v') includes only the var-slide itself. Then, all varSlides of RD_OUT_FOR(S, \(l\), \(v\)) = RF(v').

In the case that slipOf(S, \(l\)) is an If statement and \(v\) is defined in that statement, \(v'\) is an instance of a Phi var-slide. Then, RF(v') is the sum of RF(v'1) + RF(v'2) where v'1 and v'2 are the instances of v defined in both branches of the If statement. Inductively, RF(S', v'1) = the varSlides of RD_OUT_FOR(S, \(l'\)+[1], \(v\)) and RF(S', v'2) = the varSlides of RD_OUT_FOR(S, \(l'\)+[2], \(v\)), where \(l'\)+[1] and \(l'\)+[2] are the labels of both branches of the If statement. The set of RD_OUT for an If statement is the sum of the set of RD_OUT for the Then and the Else branch of the If statement (by definition of RD_OUT in Listing 2.3.1). Therefore the varSlides of RD_OUT_FOR(S, \(l'\)+[1], \(v\)) + the varSlides of RD_OUT_FOR(S, \(l'\)+[2], \(v\)) is exactly the varSlides of RD_OUT_FOR(S, \(l\), \(v\)).

**Theorem 2** (Reaching definitions and liveness for entry). Let \(S\) be a statement, and \(S'\) be its SSA form. Let \(l', v'\) be a valid label in \(S'\) and a live on entry instance of \(S'\) at \(l'\), respectively. Let \(v, l\) be the variable of \(v'\) and the corresponding label of \(l'\) using the VarLabelOf function, respectively.

We then say that all varSlides of RD_IN_FOR(S, \(l\), \(v\)) = RF(v').

**Examples**

The following examples demonstrate the theorem:

1. For the first example, let \(l', v'\) be 13 and sum5, respectively, as sum5 is a live on exit instance of \(S'\) at 13. Let \(v, l\) be sum and 11, respectively, as sum is the variable of sum5 and 11 is the label of 13 in \(S\). The set RD_OUT_FOR(S, 11, sum) is \{(sum, 2), (sum, 11)\}, and its corresponding set of varSlides is \{sum2, sum8\}. The varSlide of sum5 is Phi, therefore RF(sum5) is \{sum2, sum8\}.

2. For the second and final example, let \(l', v'\) be 14 and max6, respectively, as max6 is a live on exit instance of \(S'\) at 14. Let \(v, l\) be max and 12, respectively, as max is the variable of max6 and 12 is the label of 14 in \(S\). The set RD_OUT_FOR(S, 12, max) is \{(max, 1), (max, 7)\}, and its corresponding set of varSlides is \{max1, max7\}. The varSlide of max6 is Phi, therefore RF(max6) is \{max1, max7\}.
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The formal proof can be found in Appendix A.2.2. However, here we provide a short description of the proof.

Let \( l \) be a valid label in \( S \) (where \( l = l_1 + [c] \)), \( l' \) be the varLabel of \( l \), and slipOf(\( S, l \)) be a SeqComp statement. In the case that \( c = 1 \), the set of LV_ENTRY for the left branch of a SeqComp statement is equal to the set of LV_ENTRY of the SeqComp statement itself (by definition of LV_ENTRY in Listing 2.3.2), therefore \( v' \) is live on entry at \( l' \). Inductively, RF(\( S', v' \)) = the varSlides of RD_IN_FOR(\( S, l \)).

The set of RD_IN for a SeqComp statement is equal to the set of RD_IN for the left branch of a SeqComp statement (by definition of RD_IN in Listing 2.3.1), therefore the varSlides of RD_IN_FOR(\( S, l \)) = the varSlides of RD_IN_FOR(\( S, l \)). In the case that \( c = 2 \), the set of LV_ENTRY for the right branch of a SeqComp statement is equal to the set of LV_EXIT for the left branch of the SeqComp statement (by definition of LV_ENTRY and LV_EXIT in Listing 2.3.2), therefore \( v' \) is live on exit at \( l' + [1] \). Using Theorem 1, RF(\( S', v' \)) = the varSlides of RD_OUT_FOR(\( S, l + [1] \)). The set of RD_OUT for the left branch of a SeqComp statement is equal to the set of RD_IN for the right branch of the SeqComp statement (by definition of RD_IN and RD_OUT in Listing 2.3.1), therefore the varSlides of RD_OUT_FOR(\( S, l + [1] \)) = the varSlides of RD_IN_FOR(\( S, l \)).

3.2 Reachability in the slideDG and the varSlideDG

We now turn to describe the correspondence between slides and paths of the slide-dependence graph of \( S \), and varSlides and paths of the varSlide-dependence graph of \( S' \), using the following two theorems:

**Theorem 3** (Path correspondence). Given a statement \( S \), its slide-dependence graph slideDG, its varSlide-dependence graph varSlideDG, and two slides \( slide1 \) and \( slide2 \) where \( slide2 \) is reachable from \( slide1 \) in slideDG using \( via \), then the varSlide of \( slide2 \) is reachable from the varSlide of \( slide1 \) in varSlideDG.

**Theorem 4** (Path back correspondence). Given a statement \( S \), its slide-dependence graph slideDG, its varSlide-dependence graph varSlideDG, and two slides \( slide1 \) and \( slide2 \) where the varSlide of \( slide2 \) is reachable from the varSlide of \( slide1 \) in varSlideDG, then \( slide2 \) is reachable from \( slide1 \) in slideDG.

The formal definition and proof of Path correspondence is as follows:


- **requires** Valid(S) ∧ Core(S)
- **requires** Valid(S’) ∧ Core(S’)
- **requires** IsSlideDGOf(slideDG, S)
- **requires** IsVarSlideDGOf(varSlideDG, S)
- **requires** ValidXLS(glob(S), XLS, X)
- **requires** RemoveEmptyAssignments(Rename(S’, XLS, X, glob(S))) = S
- **requires** SlideDGReachableVia(slideDG, slide1, via, slide2, SlideDGSlides(slideDG))

- **ensures** VarSlideDGReachable(varSlideDG, VarSlideOf(S, S’, slide1, XLS, x), VarSlideOf(S, S’, slide2, XLS, x), VarSlideDGVarSlides(varSlideDG))

- **decreases** via
Proof. Let slide1 and slide2 be slides of S, and varSlide1 and varSlide2 be their corresponding varSlides of S’. We need to show that if slide2 is reachable from slide1 then varSlide2 is reachable from varSlide1. Let via be the SlideDGPath between slide1 and slide2.

Base case: If via is Empty then in fact slide1 = slide2, therefore varSlide1 = varSlide2 which means that varSlide2 is reachable from varSlide1.

via is Extend(prefix, n), then n is reachable from slide1 using the path prefix, and n is a predecessor of slide2. Let n’ be the corresponding varSlide of n. We need to show that n’ is reachable from varSlide1 and that varSlide2 is reachable from n’.

Inductive hypothesis: Suppose n’ is reachable from varSlide1.

Inductive step: n is a predecessor of slide2, therefore varSlide2 is reachable from n’ using Lemma 5 (as explained next). We have shown a path between varSlide1 and n’ and between n’ and varSlide2, therefore there is a path between varSlide1 and varSlide2 and we can say that varSlide2 is reachable from varSlide1.

\[\begin{array}{c}
\text{slide1} \\
\text{prefix} \\
\text{n} \\
\text{slide2}
\end{array}\]

\[\begin{array}{c}
\text{varSlide1} \\
\text{recursively} \\
\text{EdgeToVarPhiPath} \\
\text{n’} \\
\text{varSlide2}
\end{array}\]

**Figure 3.1: PathCorrespondence demonstration.**

**Lemma 5 (Edge to var phi path).**

\[
\text{Lemma } \text{EdgeToVarPhiPath}(\text{slide1}: \text{Slide}, \text{slide2}: \text{Slide}, \text{slideDG}: \text{SlideDG}, \text{varSlideDG}: \text{VarSlideDG}, \text{S}: \text{Statement}, \text{S’}: \text{Statement}, \text{XLs}: \text{seq<set<Variable>>, X: seq<Variable>})
\]

\[
\begin{array}{c}
\text{requires } \text{Valid}(S) \land \text{Core}(S) \\
\text{requires } \text{Valid}(S’) \land \text{Core}(S’) \\
\text{requires } \text{IsSlideDGOf}(\text{slideDG}, S) \\
\text{requires } \text{IsVarSlideDGOf}(\text{varSlideDG}, S) \\
\text{requires } \text{SlideDependence}(\text{slide1}, \text{slide2}, S) \\
\text{requires } \text{ValidXLs}(\text{glob}(S), \text{XLs}, X) \\
\text{requires } \text{RemoveEmptyAssignments}(\text{Rename}(S’, \text{XLs}, X, \text{glob}(S))) = S \\
\text{ensures } \text{VarSlideDGReachablePhi}(\text{varSlideDG}, \text{VarSlideOf}(S, S’, \text{slide1}, \text{XLs}, x), \text{VarSlideOf}(S, S’, \text{slide2}, \text{XLs}, x), \text{VarSlideDGVarSlides}(\text{varSlideDG}))
\end{array}
\]

**Proof.** Let \(v\) be the variable defined in slide1 and used in slide2. Let \(l1\) be the label of slide1 where \(v\) is defined, and \(l2\) be the label of slide2 where \(v\) is used. Let vSlide1, varL1 and vSlide2, varL2 be the corresponding var-slides and labels of slide1, l1 and slide2, l2, respectively. Let \(v’\) be the instance of \(v\) defined in vSlide1 and \(v’\) be the instance of \(v\) used in vSlide2, which is live on entry in varL2.

Using Theorem 2, we can say that all varSlides of RD_IN_FOR(S, l2, v) = RF(v’). There is a slide-dependence between slide1 and slide2, therefore (v, l1) is in RD_IN_FOR(S, l2, v) and its varSlide of vSlide1 is in RF(v’).
If $v'$ is Regular, then by definition $RF(v')$ is $\{\text{varSlide of } v'\}$. We already saw that $vSlide1$ is in $RF(v')$, therefore the varSlide of $v'$ is exactly $vSlide1$ which states that $v'' = v'$. Therefore, there is an edge between $vSlide1$ and $vSlide2$ in varSlideDG and $vSlide2$ is reachable from $vSlide1$.

If $v'$ is Phi, then by definition $RF(v')$ is the regular predecessors of $v'$. We already saw that $vSlide1$ is in $RF(v')$, therefore there is a path of 0 or more phi varSlides between the $vSlide1$ and varSlide of $v'$. Furthermore, since $v'$ is used in $vSlide2$, there is an edge between the varSlide of $v'$ and $vSlide2$. Therefore, there is a path of 0 or more phi varSlides between $vSlide1$ and $vSlide2$, and we can say that $vSlide2$ is reachable from $vSlide1$.

![Figure 3.2: EdgeToVarPhiPath example: Slide 1 as slide1, slide 13 as slide2, max1 as $v''$ and max6 as $v'$.

The formal definition and proof of Path back correspondence is as following:

**lemma** PathBackCorrespondence (slideDG: SlideDG, varSlideDG: VarSlideDG, slide1: Slide, slide2: Slide, via: VarSlideDGPath, S: Statement, S': Statement, XLs: seq<set<Variable>>, X: seq<Variable>)

- **requires** Valid(S) ∧ Core(S)
- **requires** Valid(S') ∧ Core(S')
- **requires** IsSlideDGOf(slideDG, S)
- **requires** IsVarSlideDGOf(varSlideDG, S)
- **requires** ValidXLs(glob(S), XLs, X)
- **requires** RemoveEmptyAssignments(Rename(S', XLs, X, glob(S))) = S
- **requires** VarSlideTag(VarSlideOf(S, S', slide1, XLs, x)) = Regular ∧ VarSlideTag(VarSlideOf(S, S', slide2, XLs, x)) = Regular
- **requires** VarSlideDGReachableVia(varSlideDG, VarSlideOf(S, S', slide1, XLs, x), via, VarSlideOf(S, S', slide2, XLs, x), VarSlideDGVarSlides(varSlideDG))

- **ensures** SlideDGReachable(slideDG, slide1, slide2, SlideDGSlides(slideDG))

- **decreases** via

**Proof.** Let slide1 and slide2 be slides of S, and varSlide1 and varSlide2 be their corresponding varsides of S'. We need to show that if varSlide2 is reachable from
varSlide1 then slide2 is reachable from slide1. Let via be the VarSlideDGPath between varSlide1 and varSlide2.

Base case: If via is Empty then in fact varSlide1 = varSlide2, therefore slide1 = slide2 which means that slide2 is reachable from slide1.

via is Extend(prefix, n), then n is reachable from varSlide1 using the path prefix, and n is a predecessor of varSlide2. Let n'' be the last regular varSlide on the VarSlideDGPath between varSlide1 and varSlide2 (could be n or another varSlide if n is Phi), and let prefix'' be the VarSlideDGPath that is prefix without n (meaning up until n''). Let n be the corresponding slide of n''. We need to show that n is reachable from slide1 and that slide2 is reachable from n.

Inductive hypothesis: Suppose n is reachable from slide1.

Inductive step: varSlide2 is reachable from n'', therefore slide2 is reachable from n using Lemma 6 (as explained next). We have shown a path between slide1 and n and between n and slide2, therefore there is a path between slide1 and slide2 and we can say that slide2 is reachable from slide1.

\[\text{Figure 3.3: PathBackCorrespondence demonstration.}\]

**Lemma 6** (Var phi path to edge).

**Lemma** VarPhiPathToEdge(slide1: Slide, slide2: Slide, slideDG: SlideDG, varSlideDG: VarSlideDG, S: Statement, S': Statement, X: seq<set<Variable>>, XLs: seq<Variable>)

requires Valid(S) ∧ Core(S)
requires Valid(S') ∧ Core(S')
requires IsSlideDGOf(slideDG, S)
requires IsVarSlideDGOf(varSlideDG, S)
requires VarSlideTag(VarSlideOf(S, S', slide1, XLs, x)) = Regular
requires VarSlideTag(VarSlideOf(S, S', slide2, XLs, x)) = Regular
requires VarSlideDGReachablePhi(varSlideDG, VarSlideOf(S, S', slide1, XLs, x), VarSlideOf(S, S', slide2, XLs, x), VarSlideDGVarSlides(varSlideDG))
requires ValidXLs(glob(S), XLs, X)
requires RemoveEmptyAssignments(Rename(S', XLs, X, glob(S))) = S
ensures SlideDependence(slide1, slide2, S)
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Proof. Let $v_{\text{Slide1}}$, $\text{varL1}$ and $v_{\text{Slide2}}$, $\text{varL2}$ be the corresponding var-slides and labels of slide1, l1 and slide2, l2, respectively. Let $v$ be the variable of the instance defined in $v_{\text{Slide1}}$ and used in $v_{\text{Slide2}}$. Let $v''$ be the instance of $v$ defined in $v_{\text{Slide1}}$ and $v'$ be the instance of $v$ used in $v_{\text{Slide2}}$, which is live on entry in $\text{varL2}$.

If $v'' = v'$, then there is an edge between $v_{\text{Slide1}}$ and $v_{\text{Slide2}}$ in varSlideDG and an edge between slide1 and slide2 in slideDG, meaning slide2 is slide-dependent on slide1.

If $v'' \neq v'$, then $v'$ is Phi (recall that $v'$ is live on entry in $\text{varL2}$). Using Theorem 2, we can say that all varSlides of $\text{RD\_IN\_FOR}(S, l2, v) = \text{RF}(v')$. $v''$ is a regular predecessor of $v'$, meaning it is in $\text{RF}(v')$, therefore its slide ($v$, l1) is in $\text{RD\_IN\_FOR}(S, l2, v)$. Thus, slide2 is slide-dependent on slide1.

\[\square\]

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig3_4.png}
\caption{VarPhiPathToEdge example: max1 as $v''$, max6 as $v'$, slide 1 as slide1 and slide 13 as slide2.}
\end{figure}
Chapter 4

A novel slide-based slicing algorithm

In this section we describe our slide-based slicing algorithm and provide examples.

4.1 The algorithm

Given a statement $S$, a set of variables $V$, and a slide-dependence graph $\text{slideDG}$ of $S$, our algorithm computes the set of slides whose union is the program’s slice of $S$ for the values of each variable in $V$ when exiting $S$.

Let $S_V$ be an initial set of final-def slides of $S$ on $V$, and let $\text{WorkSet}$ be a temporary initialized to $S_V$. $\text{Poll}$ returns a slide and removes it from $\text{WorkSet}$, and for each such slide in $\text{WorkSet}$ we find its slide-dependence predecessors in the slideDG (that are not in $S_V$) and add them to $S_V$ and to $\text{WorkSet}$. The algorithm terminates when there are no slides left in $\text{WorkSet}$, and it returns $S_V$.

The algorithm has a linear time-complexity in the size of the slide-dependence graph of $S$. Let $N$ be the number of nodes and $E$ be the number of edges in the graph, in the worst-case $\text{WorkSet}$ contains all slides ($N$) and the algorithm finds all predecessors of all slides ($E$), therefore the algorithm’s worst-case time complexity is $O(N+E)$.

The algorithm is presented as Algorithm 1:

**Algorithm 1:** Slide-based slicing algorithm

```
Result: $S_V$
$S_V := \bigcup_{v \in V} \{\text{final-def slides of } S\text{ on } v\text{ in } \text{slideDG}\}$;
$\text{WorkSet} := S_V$;
while $\text{WorkSet} \neq \emptyset$ do
    $S_n := \text{Poll}(\text{WorkSet})$;
    $\text{NewlyReachable} := \{\text{Predecessors of } S_n\text{ in } \text{slideDG}\} - S_V - \{S_n\}$;
    $S_V := S_V \cup \text{NewlyReachable}$;
    $\text{WorkSet} := \text{WorkSet} \cup \text{NewlyReachable}$;
end
```

4.2 Examples

In the first example our program computes the sum of two variables (see Listing 4.1), and $V = \{w\}$ where $V$ contains a variable that is not included in the code. $S_V$ is empty (since there are no final-def slides of $S$ on $w$), therefore $\text{WorkSet}$ is empty and the loop will not execute (see Listing 4.2).
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1 \( x := 5; \)
2 \( y := 4; \)
3 \( z := x + y \)

LISTING 4.1: Program S1

In the second example our program calculates the sum and product for an array of numbers (see Listing 4.3) and our set of variables is \( V = \{ \text{sum} \} \). \( S_V \) and \( \text{WorkSet} \) will initially contain the final-def slides of \( \text{sum} \): slides 2 and 5. Then, we only add the predecessors of slide 5 to \( S_V \) (there are no predecessors for slide 2): slides 1 and 7 (see the slideDG in Figure 4.2 for the predecessors). Again, there are no predecessors for slides 1 and 7, therefore the slice of \( \text{sum} \) is \{1,2,5,7\} (see Listing 4.4).

1 \( i := 0; \)
2 \( \text{sum} := 0; \)
3 \( \text{prod} := 1; \)
4 \( \text{while } i < a.\text{length} \text{ do} \)
5 \( \text{sum} := \text{sum} + a[i]; \)
6 \( \text{prod} := \text{prod} \ast a[i]; \)
7 \( i := i + 1 \)
8 \( \text{od} \)

LISTING 4.3: Program S2

1 \( i := 0; \)
2 \( \text{sum} := 0; \)
3 \( \text{while } i < a.\text{length} \text{ do} \)
4 \( \text{sum} := \text{sum} + a[i]; \)
5 \( i := i + 1 \)
8 \( \text{od} \)

LISTING 4.4: ComputeSlice(S2,\{\text{sum}\})
In the third and final example, our program sums the values of an array in even indices or in odd indices, depending on the parity of the array’s length, respectively (see Listing 4.5). Let $V = \{\text{sum}\}$, therefore all of the programs slides $\{1,3,5,8,9\}$ will be in the resulting slice (see Listing 4.6).

```
1  sum := 0;
2  if a.length % 2 == 0 then
3     i := 0
4  else
5     i := 1
6  fi;
7  while i < a.length do
8     sum := sum + a[i];
9     i := i + 2
10  od
```

**LISTING 4.5: Program S3**

```
1  sum := 0;
2  if a.length % 2 == 0 then
3     i := 0
4  else
5     i := 1
6  fi;
7  while i < a.length do
8     sum := sum + a[i];
9     i := i + 2
10  od
```

**LISTING 4.6: ComputeSlice(S3,\{sum\})**

---

**FIGURE 4.3:** The slideDG of S3 (from Listing 4.5) and the slideDG of its slice on $\{\text{sum}\}$ (from Listing 4.6) are the same.
Chapter 5

Proof of correctness

In this section we provide the correctness proof of our slide-based slicing algorithm.

5.1 Our algorithm

The SSA-based algorithm is semantics-preserving, as proven in [7]. Our goal is to prove that the resulting statements of the two algorithms are textually identical, which means that our algorithm also results in a semantics-preserving slice. To prove this we use the following theorem:

Theorem 7 (Identical slices). Given three statements $S$, $S_1$, and $S_2$, where $S_1$ is the slide-based slice of $S$, $S_2$ is the SSA-based slice of $S$, and there are no self or empty assignments in $S$, then $S_1$ is textually identical to $S_2$.

Proof. Instead of using $S_1$ and $S_2$ we can use $\text{slipOf}(S_1, [])$ and $\text{slipOf}(S_2, [])$, therefore we apply Lemma 10 which ensures that $\text{slipOf}(S_1, []) = \text{slipOf}(S_2, [])$. The proof of Lemma 10 is provided later in this chapter.

The following terms, definitions (10, 11), and lemmas (8, 9) are needed for the proof of Lemma 10.

We begin with presenting the relevant terminology:

- $S$: The original statement.
- $V$: The set of variables for the slice.
- $S'$: The SSA version of $S$.
- $V'$: The set of the live-on-exit instance of each variable in $V$.
- $SV'$: The result of $\text{ComputeFISlice}(S', V')$.
- $res$: The SSA-based slice of $S$ on $V$.
- $SV$: The slide-based slice of $S$ on $V$.
- $slidesSV$: The set of slides of $SV$.
- $varSlidesSV$: The set of varSlides of $SV'$.

We now continue with two formal definitions:

Definition 10 (slidesSV). Given a statement $S$, a set of variables $V$, and a slide-dependence graph slideDG:
∀ Sm • Sm in SlideDGSlides(slideDG) ∧ (∃ Sn • Sn in finalDefSlides(S, slideDG, V) ∧ SlideDGReachable(slideDG, Sm, Sn, SlideDGSlides(slideDG)))

The set of slides of S in which each slide has a path to a final-def slide in the slideDG of S. In our example, V = {isSorted} then slidesSV consists of slides 1, 3, 4, 7, 13, 17, 20 and 22 (where 20 and 22 are the final-def slides).

**Definition 11 (varSlidesSV).** Given a statement S, a set of instances V', and a varSlide-dependence graph varSlideDG:

∀ Sm' • Sm' in VarSlideDGVarSlides(varSlideDG) ∧ (∃ Sn' • VarSlideVariable(Sn') in V' ∧ VarSlideDGReachable(varSlideDG, Sm', Sn', VarSlideDGVarSlides(varSlideDG)))

The set of varSlides of S' in which each varSlide has a path to a live-on-exit instance (in V') in the varSlideDG of S'. In our example, V' = {isSorted12} then varSlidesSV consists the varSlides of max1, i3, count4, max5, i5, count5, max6, max7, count9, count10, i11, isSorted12, isSorted13 and isSorted14.

We show the correspondence between slidesSV and varSlidesSV with the following lemma:

**Lemma 8 (LemmaSlidesSVToVarSlidesSV).** Given a statement S, its corresponding SSA version S', and the mapping between X and XLs, for each slide in the slideDG of S we have:

slide in slidesSV ⇐⇒ VarSlideOf(S, S', slide, XLs, X) in varSlidesSV

**Proof.** We prove the first direction of the lemma:

slide in slidesSV ⇒ VarSlideOf(S, S', slide, XLs, X) in varSlidesSV

For each slide ∈ slidesSV, we find its corresponding varSlide in S', and show that it is also in varSlidesSV. Let vSlide be the corresponding varSlide of slide using the function call vSlide := VarSlideOf(S, S', slide, XLs, X). This ensures that for each slide there is exactly one corresponding varSlide (and backwards). In order to prove that vSlide ∈ varSlidesSV we need to show that there exists an instance in V' whose varSlide is reachable from vSlide (Definition 11). Let:

- **finalDefSlide:** A final-def slide that is reachable from slide (could be that slide is in fact finalDefSlide). The final-def slides in our example (where V = {isSorted}) are slides 20 and 22.
- **finalDefVarSlide:** The varSlide of finalDefSlide. The final-def varSlides in our example are the varSlides of isSorted13 and isSorted14.
- **liveExitVarSlide:** The varSlide of v' (where v' is the instance of the variable of finalDefSlide that is in V'), which we need to prove is reachable from vSlide. Note that v' is a live on exit instance of S'. In our example it is the varSlide of isSorted12.

Using Theorem 1 where varLabel is [] (because v' is live on exit from S'), we can say that all varSlides of RD_OUT_FOR(S, [], v) are RF(v'). Using Definition 7, these varSlides are the corresponding varSlides of the final-def slides.
If \( lv\text{ExitVarSlide} \) is Regular, then \( RF(v’) \) is \{lv\text{ExitVarSlide}\} and \{finalDefVarSlide\} is in \{lv\text{ExitVarSlide}\}.

If \( lv\text{ExitVarSlide} \) is Phi, then \( RF(v’) \) is the set of regular predecessors of \( v’ \), which includes \( final\text{DefVarSlide} \). Thus, we can conclude that there is a path from \( final\text{DefVarSlide} \) to \( lv\text{ExitVarSlide} \), therefore \( lv\text{ExitVarSlide} \) is reachable from \( final\text{DefVarSlide} \).

In our example, the instance of \( lv\text{ExitVarSlide} \) is \( \text{isSorted}12 \) and the set reaching out from the corresponding point is \( S \) is \{(isSorted,20), (isSorted,22)\}. Using Theorem 1, the varSlides of this set are \{isSorted13, isSorted14\} and are the regular-predecessor varSlides of the varSlide of \( \text{isSorted}12 \).

Now we prove the second direction of the lemma:

\[
\text{VarSlideOf}(S, S', \text{slide}, \text{XLS}, X) \in \text{varSlidesSV} \implies \text{slide} \in \text{slidesSV}
\]
or in other words:

\[
\text{slide} \not\in \text{slidesSV} \implies \text{VarSlideOf}(S, S', \text{slide}, \text{XLS}, X) \not\in \text{varSlidesSV}
\]

Suppose \( \text{slide} \not\in \text{slidesSV} \), let us assume for the sake of contradiction that \( v\text{Slide} \in \text{varSlidesSV} \). By Definition 10, there exists a varSlide, \( lv\text{ExitVarSlide} \), whose variable is in \( V' \) and is reachable from \( v\text{Slide} \). Recall that \( v\text{Slide} \) is a regular varSlide (formed from the \text{VarSlideOf} function). Let \( Sn' \) be the last regular varSlide on a VarSlideDG-Path between \( v\text{Slide} \) and \( lv\text{ExitVarSlide} \) (including \( lv\text{ExitVarSlide} \), and let \( Sn \) be its corresponding slide. Again, the instance of \( lv\text{ExitVarSlide} \) is live-on-exit from \( S' \), and by using Definition 7 the set RD\_OUT(\( S, [] \)) from the corresponding label \( [] \) in \( S \) are final-def slides of \( S \). By Theorem 1, the varSlides of this set are regular-predecessors of \( lv\text{ExitVarSlide} \), therefore \( Sn \) is a final-def slide of \( S \). By using Theorem 4 (PathBack-Correspondence, as explained in Chapter 3) \( Sn \) is reachable from \( \text{slide} \). According to Definition 10, \( \text{slide} \in \text{slidesSV} \), in contradiction to \( \text{slide} \not\in \text{slidesSV} \), hence \( v\text{Slide} \not\in \text{varSlidesSV} \).

We have proved that each \( \text{slide} \) in \( \text{slidesSV} \) has a corresponding \( v\text{Slide} \) in \( \text{varSlidesSV} \).

We also gained another property: \( |\text{slidesSV}| = |\text{varSlidesSV}| \), by counting exactly one varSlide for each slide using the \text{VarSlideOf} function.

Next, we define the following predicate:

\[
\text{predicate MatchingSlips}(S1: \text{Statement}, S2: \text{Statement}, l: \text{Label})
\]
reads *
requires Valid(S1) \land Valid(S2)
requires Core(S1) \land Core(S2)
requires ValidLabel(l, S1)
requires ValidLabel(l, S2)
{
var slipOfS1 := slipOf(S1, l);
var slipOfS2 := slipOf(S2, l);

match slipOfS1 |
case Skip \Rightarrow IsSkip(slipOfS2)
case Assignment(LHS,RHS) \Rightarrow IsSkip(slipOfS2) \lor IsAssignment(slipOfS2)
case SeqComp(S1,S2) \Rightarrow IsSkip(slipOfS2) \lor IsSeqComp(slipOfS2)
case IF(B,Sthen,Selse) \Rightarrow
\]
Chapter 5. Proof of correctness

\[ \text{IsSkip}(\text{slipOfS2}) \lor \text{IsIF}(\text{slipOfS2}) \]
\[ \text{case } \text{DO}(B, \text{Sloop}) \Rightarrow \]
\[ \text{IsSkip}(\text{slipOfS2}) \lor \text{IsDO}(\text{slipOfS2}) \]
\]

We use this predicate for \( S \) and \( \text{res} \) in order to prove the following lemma:

**Lemma 9 (LemmaMatchingSlips).** Given a statement \( S \), its SSA-based slice \( \text{res} \), and a valid label in both statements \( l \):

\( \text{MatchingSlips}(S, \text{res}, l) \)

**Proof.** Let \( \text{slipOfS} \) be the result of \( \text{slipOf}(S, l) \), and \( \text{slipOfRes} \) the result of \( \text{slipOf}(\text{res}, l) \). For this proof we distinguish between the various values of \( \text{slipOfS} \).

- If \( \text{slipOfS} \) is Skip, then by converting Skip to SSA, computing its FI-Slice and converting back to SSA - it will stay Skip, therefore \( \text{slipOfRes} \) is also Skip.

- If \( \text{slipOfS} \) is Assignment, by converting it to SSA it will stay an Assignment. After computing its FI-Slice it will stay Assignment or become Skip (if it is not in the slice). Finally, the result of converting back from SSA will be Assignment or Skip.

- If \( \text{slipOfS} \) is Sequential Composition, by converting it to SSA it will stay Sequential Composition. After computing its FI-Slice it will stay Sequential Composition or become Skip (if it is not in the slice). Finally, the result of converting back from SSA will be either Sequential Composition or Skip.

- If \( \text{slipOfS} \) is IF, by converting it to SSA it will stay as IF. After computing its FI-Slice it will stay IF or become Skip (if it is not in the slice). Finally, the result of converting back from SSA will be either IF or Skip.

- If \( \text{slipOfS} \) is DO, by converting it to SSA it will be Sequential Composition of Phi-Assignment and DO. After computing its FI-Slice the Sequential Composition will stay Sequential Composition or become Skip (if it is not in the slice). Finally, the result of converting back from SSA will return to be either DO or Skip.

Finally, in order to show that \( \text{slipOf}(SV, []) = \text{slipOf}(\text{res}, []) \) (and complete the proof of Theorem 7) we need to prove the following lemma:

**Lemma 10 (LemmIdenticalSlips).** Given the slide-based algorithm slice \( SV \), the SSA-based algorithm slice \( \text{res} \) (both for a statement \( S \) and a set of variables \( V \)), and a valid label in both slices \( l \):

\( \text{slipOf}(SV, l) = \text{slipOf}(\text{res}, l) \)

**Proof.** We start by using Lemma 11 (full definition and proof in Appendix A.2.3), which states that: \( \text{IsSkip}(\text{slipOf}(SV, l)) \iff \text{IsSkip}(\text{slipOf}(\text{res}, l)) \). If \( \text{slipOf}(SV, l) \) is Skip, then \( \text{slipOf}(\text{res}, l) \) is Skip; if \( \text{slipOf}(SV, l) \) is not Skip, then \( \text{slipOf}(SV, l) \) is also not Skip (\( SV \) is a substatement of \( S \)) and we use Lemma 9 to determine that \( \text{slipOf}(SV, l) \) and \( \text{slipOf}(\text{res}, l) \) are of the same type. We prove this lemma by induction on \( l \):

**Base case:**
• slipOf(SV, l) is Skip: slipOf(res, l) is also Skip, as explained using Lemma 11.

• slipOf(SV, l) is Assignment: slipOf(res, l) is also Assignment (as explained above) and we need to show that it is exactly the same assignment. First, we find the set of slides of all the assignments in slipOf(SV, l) and the set of varSlides of all the assignments in slipOf(SV', l') (where l' is the corresponding label of l in SV') and mark them as A and B, respectively. We already know that \(|\text{slides}_{SV}| = |\text{varSlides}_{SV}|\) for each slide in \(\text{slides}_{SV}\) has a corresponding \(\text{varSlide}\) in \(\text{varSlides}_{SV}\) (and backwards). We also know that A is a subset of \(\text{slides}_{SV}\) and B is a subset of \(\text{varSlides}\), therefore \(|A| = |B|\). Then, for each slide and variable in the first set, there is a corresponding varSlide and instance in the second set (Lemma 8), and if we convert each of those instances back to their original variables we will get the assignment of res (which is the exact same assignment of SV).

Inductive hypothesis:

• If slipOf(SV, l) is a Sequential Composition statement, suppose slipOf(SV, l+[1]) = slipOf(res, l+[1]) and slipOf(SV, l+[2]) = slipOf(res, l+[2]) (both branches of the Sequential Composition statement).

• If slipOf(SV, l) is an If statement, suppose slipOf(SV, l+[1]) = slipOf(res, l+[1]) and slipOf(SV, l+[2]) = slipOf(res, l+[2]) (both branches of the If statement).

• If slipOf(SV, l) is a Do statement, suppose slipOf(SV, l+[1]) = slipOf(res, l+[1]) (loop body of the Do statement).

Inductive step:

• If slipOf(SV, l) is a Sequential Composition statement, slipOf(res, l) is of the same type (as explained above). By our hypothesis, slipOf(SV, l) = slipOf(res, l).

• If slipOf(SV, l) is an If statement, slipOf(res, l) is of the same type (as explained above). Let \(b\) be the boolean expression in slipOf(S, l). In the transition to SSA, the variables in \(b\) were renamed into new instances in slipOf(S', l'), where l' is the result of VarLabelOf function on l. Then, after computing its flow-insensitive slice, \(b\) remained the same exact expression in slipOf(SV', l'). In the transition back from SSA, the instances in \(b\) were renamed to their original variables in slipOf(res, l), where l is the result of LabelOf function on l' (and using Lemma 12, in Appendix A.2.3), it is the original l. Therefore \(b\) remained the same exact boolean expression in slipOf(res, l) as in slipOf(S, l). By our hypothesis, slipOf(SV, l) = slipOf(res, l).

• If slipOf(SV, l) is a Do statement, slipOf(res, l) is of the same type (as explained above). The boolean expression in slipOf(res, l) is the same exact boolean expression in slipOf(S, l), as previously explained. By our hypothesis, slipOf(SV, l) = slipOf(res, l).

\(\square\)

To summarize, we proved that our slide-based algorithm is semantics-preserving. This was done by showing that the SSA-based slice of S is textually identical to the slide-based slice of S.
Chapter 6

Conclusion

In this thesis we have presented a new slide-based slicing algorithm. We proved that the resulting slice of the algorithm is textually identical to the resulting slice of an existing SSA-based slicing algorithm [7], which is semantics-preserving, therefore our algorithm is also semantics-preserving. Our algorithm is syntax-preserving, therefore we gained syntax-preservation for the SSA-based algorithm. We have also discussed how we improved the complexity of the SSA-based algorithm using our algorithm.

We have formalized the slide-dependence graph used in our slicing algorithm, and formalized the varSlide-dependence graph and the connection between the two graphs in order to prove that both resulting slices are textually identical.

We have also formalized the transition of a program into SSA, the computation of its flow-insensitive slice, and the transition back from SSA. Then we showed that the resulting slice of every slicing algorithm that meets these requirements is syntax-preserving and textually identical to our algorithms resulting slice.

In this thesis we have made some simplifying assumptions. Our algorithm computes a slice for a statement of a rather simple language and does not support complex statements such as goto or switch statements, as we wanted to prove the textual equivalence to the code in [7]. Moreover, we can use slideDG or PDG as a program representation also for complex statements (as been done in [4] and [8]) and forgoing the semantic-preservation of the statements. Also, some of the lemmas needed in the proof of our algorithm have been used with the Dafny language without providing a formal proof, due to the scope of this thesis.

Some directions for further research are:

- Implementing a co-slicing algorithm (described in [7]), an advanced sliding transformation in which the complement reuses a selection of extracted results, thus yielding a potentially smaller complement.

- Using our formal framework in order to prove the correctness of a PDG-based slicing algorithm.

- Expanding our algorithm for slicing from a different point in the statement. For a statement $S$ and a set of variables $V$, our algorithm uses backward slicing from the final-def nodes of each variable in $V$. This could be expanded by slicing from any regular node and each variable, not necessarily its final-def.

- Using our formal framework to develop algorithms for cloning, code-motion refactoring, etc.
We begun our work by developing a slicing algorithm that is using slide-dependence graph for program representation. In order to prove our algorithm we have developed a solid formal framework that, as described above, can be used for many other applications that improve code quality.
Appendix A

Appendix

A.1 Full definitions

A.1.1 Utility functions

**predicate** Valid(S: Statement)
[
  match S {
    case Skip ⇒ true
    case Assignment(LHS, RHS) ⇒ ValidAssignment(LHS, RHS)
    case SeqComp(S1, S2) ⇒ Valid(S1) ∧ Valid(S2)
    case IF(B0, Sthen, Selse) ⇒
      (∀ state: State • B0.0. requires(state)) ∧ Valid(Sthen) ∧ Valid(Selse)
    case DO(B, Sloop) ⇒
      (∀ state: State • B.0. requires(state)) ∧ Valid(Sloop)
      ∧ ∀ state1: State, P: Predicate • P.0. requires(state1)
  }
]

**predicate** Core(stmt: Statement)
[
  match stmt {
    case Skip ⇒ true
    case Assignment(LHS, RHS) ⇒ true
    case SeqComp(S1, S2) ⇒ Core(S1) ∧ Core(S2)
    case IF(B0, Sthen, Selse) ⇒ Core(Sthen) ∧ Core(Selse)
    case DO(B, Sloop) ⇒ Core(Sloop)
  }
]

**function** method setOf<T>(s: seq<T>): (res: set<T>)
ensures ∀ v • v in res ⇐⇒ v in s
[
  set x | x in s
]

**predicate** ValidAssignment(LHS: seq<Variable>, RHS: seq<Expression>)
[
  |LHS| = |RHS| ∧ |setOf(LHS)| = |LHS|
]

**predicate** ValidLabel(l: Label, S: Statement)
reads *
requires Valid(S) ∧ Core(S)
[
  if l = [] then true
  else
    match S {

Appendix A. Appendix

\textbf{function method} \textbf{def}(S: Statement): set\langle\text{Variable}\rangle

\begin{verbatim}
match S {
case Assignment(LHS,RHS) \Rightarrow setOf(LHS)
case Skip \Rightarrow {}
case SeqComp(S1,S2) \Rightarrow def(S1) + def(S2)
case IF(B0,Sthen,Selse) \Rightarrow def(Sthen) + def(Selse)
case DO(B,Sloop) \Rightarrow def(Sloop)
}
\}
\end{verbatim}

\textbf{function method} \textbf{glob}(S: Statement): set\langle\text{Variable}\rangle

\begin{verbatim}
set x | x in def(S) + input(S)
\end{verbatim}

\textbf{function} \textbf{GetRHSVariables}(seqExp: seq\langle\text{Expression}\rangle): set\langle\text{Variable}\rangle

\begin{verbatim}
if seqExp = [] then {}
else seqExp[0].1 + GetRHSVariables(seqExp[1..])
\end{verbatim}

\textbf{A.1.2 Slides functions}

\textbf{function} \textbf{SlideLabel}(s: Slide): Label \{ s.0 \}

\textbf{function} \textbf{SlideVariable}(s: Slide): Variable \{ s.1 \}

\textbf{function} \textbf{SlideLabels}(s: Slide, S: Statement): set\langle\text{Label}\rangle

\begin{verbatim}
requires Valid(S) \land Core(S)
requires s in SlidesOf(S, def(S))
\begin{verbatim}
set l | l = SlideLabel(s) \lor
(1 < SlideLabel(s) \land (IsDO(slipOf(S,1)) \lor IsIF(slipOf(S,1))))
\end{verbatim}
\end{verbatim}

\textbf{function} \textbf{SlidesOf}(S: Statement, V: set\langle\text{Variable}\rangle)

\begin{verbatim}
: set\langle\text{Slide}\rangle
\begin{verbatim}
requires Valid(S) \land Core(S)
\end{verbatim}
\begin{verbatim}
ensures \forall s \bullet s in SlidesOf(S, V) \implies
ValidLabel(SlideLabel(s), S) \land
\neg IsEmptyAssignment(slipOf(S, SlideLabel(s)))
\end{verbatim}
\begin{verbatim}
SlidesOfRec(S, V, [])
\end{verbatim}
\end{verbatim}

\textbf{function} \textbf{SlidesOfRec}(S: Statement, V: set\langle\text{Variable}\rangle,
Appendix A. Appendix

1: Label): (slides: set<Slide>)

reads *

requires Valid(S) ∧ requires Core(S)

ensures ∀ s • s in slides ⇒

ValidLabel(SlideLabel(s), S) ∧
¬IsEmptyAssignment(slipOf(S, SlideLabel(s)))

match S {

case Skip ⇒ {}

case Assignment(LHS,RHS) ⇒

set v | v in V * setOf(LHS) • (1, v)

case SeqComp(S1,S2) ⇒

SlidesOfRec(S1, V, 1+[1]) + SlidesOfRec(S2, V, 1+[2])

case IF(B0,Sthen,Selse) ⇒

SlidesOfRec(Sthen, V, 1+[1]) + SlidesOfRec(Selse, V, 1+[2])

case DO(B,Sloop) ⇒

SlidesOfRec(Sloop, V, 1+[1])
}

}

function UsedVars(S: Statement, l: Label): set<Variable>

requires Valid(S) ∧ Core(S)

requires ValidLabel(1, S)

var slipOfS := slipOf(S, 1);

match S {

case Assignment(LHS,RHS) ⇒ set v | v in GetRHSVariables(RHS)

case SeqComp(S1,S2) ⇒ []

case IF(B,Sthen,Selse) ⇒ set v | v in B.1

case DO(B,S0) ⇒ set v | v in B.1

case Skip ⇒ []
}

}

function GetRHSVariables(seqExp: seq<Expression>): set<Variable>

if seqExp = [] then {}
else seqExp[0].1 + GetRHSVariables(seqExp[1..])

}

function SlideDGStatement(slideDG: SlideDG): Statement | slideDG.0 |

function SlideDGSlides(slideDG: SlideDG): set<Slide> | slideDG.1 |

function SlideDGMap(slideDG: SlideDG): map<Slide, set<Slide>> | slideDG.2 |

A.1.3 VarSlides functions

function VarSlideVariable(s: VarSlide): Variable | s.0 |

function VarSlideTag(s: VarSlide): VarSlideTag | s.1 |

function VarSlideLabels(s: VarSlide, S: Statement): set<Label>

requires Valid(S) ∧ Core(S)

requires s in VarSlidesOf(S, def(S))

var assignmentLabels := VarSlideAssignmentLabels(s, S, []);

assignmentLabels +

set 1 | ((∀ 1’ • 1’ in assignmentLabels ⇒ 1 < 1’) ∧
(IsDO(slipOf(S, l)) ∨ IsIF(slipOf(S, l)))

function VarSlideAssignmentLabels(s: VarSlide, S: Statement, l: Label): set<Label>
reads *
requires Valid(S) ∧ Core(S)
requires ValidLabel(1, S)
requires s in VarSlidesOf(S, def(S))
{
  match S {
    case Assignment(LHS, RHS) ⇒
      if VarSlideVariable(s) in LHS then {l} else {}
    case Skip ⇒ {} 
    case SeqComp(S1, S2) ⇒
      VarSlideAssignmentLabels(s, S1, l + [1]) +
      VarSlideAssignmentLabels(s, S2, l + [2])
    case IF (B0, Sthen, Selse) ⇒
      VarSlideAssignmentLabels(s, Sthen, l + [1]) +
      VarSlideAssignmentLabels(s, Selse, l + [2])
    case DO(B, Sloop) ⇒
      VarSlideAssignmentLabels(s, Sloop, l + [1])
  }
}

function VarSlidesOf(S’: Statement, V: set<Variable>):
set<VarSlide>
reads *
requires Valid(S’) ∧ Core(S’)
{
  match S’ {
    case Skip ⇒ {} 
    case Assignment(LHS, RHS) ⇒
      set v | v in setOf(LHS) • (v, Regular)
    case SeqComp(S1, S2) ⇒
      if IsDO(S2) then
        assert IsAssignment(S1);
      match S1
        case Assignment(LHS, RHS) ⇒
          (set v | v in setOf(LHS) • (v, Phi)) + VarSlidesOf(S2, V)
      else
          VarSlidesOf(S1, V) + VarSlidesOf(S2, V)
    case IF (B0, Sthen, Selse) ⇒
      assert IsSeqComp(Sthen);
      match Sthen
        case SeqComp(S1, S2) ⇒
          assert IsAssignment(S2);
          match S2
            case Assignment(LHS, RHS) ⇒
              ((set v | v in setOf(LHS) • (v, Phi)) + VarSlidesOf(S1, V) +
              assert IsSeqComp(Selse);
              match Selse
                case SeqComp(S1’, S2’) ⇒
                  assert IsAssignment(S2’);
                  match S2’
                    case Assignment(LHS’, RHS’) ⇒
                      (set v | v in setOf(LHS’) • (v, Phi)) + VarSlidesOf(S1’, V))
      case DO(B, Sloop) ⇒
        assert IsSeqComp(Sloop);
      match Sloop
        case SeqComp(S1, S2) ⇒
          assert IsAssignment(S2);
          match S2

Appendix A. Appendix

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A.1.4 Graphs correspondence functions

function VarLabelOf(S: Statement, S': Statement, l: Label, XLS: seq<set<Variable>>, X: seq<Variable>): Label
reads *
requires Valid(S) \land Valid(S')
requires Core(S) \land Core(S')
requires ValidLabel(l, S)
requires ValidXLS(glob(S), XLS, X)
requires S = RemoveEmptyAssignments(Rename(S', XLS, X, glob(S)))
requires MatchingSlipsToSSA(S, [], S', [])
ensures ValidLabel(VarLabelOf(S, S', l, XLS, X), S')
{
    match S {
    case Skip ⇒
        assert IsSkip(S');
        assert l = [ ];
        []
    case Assignment(LHS, RHS) ⇒
        assert IsAssignment(S');
        assert l = [ ];
        []
    case SeqComp(S1, S2) ⇒
        assert IsSeqComp(S');
        if l = [ ] then [] else
            match S' {
                case SeqComp(S1', S2') ⇒
                    if l[0] = 1 then [1] + VarLabelOf(S1, S1', 1[1..], XLS, X)
                    else [2] + VarLabelOf(S2, S2', 1[1..], XLS, X)
            }
    case IF(B0, Sthen, Selse) ⇒
        assert IsIF(S');
        if l = [ ] then [] else
            match S' {
                case If(B0', Sthen', Selse') ⇒
                    if l[0] = 1 then [1,1] + VarLabelOf(Sthen, Sthen', 1[1..], XLS, X)
                    else [2,1] + VarLabelOf(Selse, Selse', 1[1..], XLS, X)
            }
    case DO(B, Sloop) ⇒
        assert IsDO(S');
        if l = [ ] then [] else
            match S' {
                case SeqComp(S1', S2') ⇒
                    assert IsDO(S2');
                    match S2' {
                        case DO(B', Sloop') ⇒
                            [2,1,1] + VarLabelOf(Sloop, Sloop', 1[1..], XLS, X)
                    }
                }
            }
    }
}

function VarSlideDGStatement(varSlideDG: VarSlideDG): Statement
{
    varSlideDG.0
}

function VarSlideDGVarSlides(varSlideDG: VarSlideDG): set<VarSlide> {
    varSlideDG.1
}

function VarSlideDGMap(varSlideDG: VarSlideDG): map<VarSlide, set<VarSlide>> {
    varSlideDG.2
}
predicate MatchingSlipsToSSA (S: Statement, l: Label, 
S': Statement, l': Label) 
reads *
requires Valid(S) \∧ Valid(S')
requires Core(S) \∧ Core(S')
requires ValidLabel(1, S)
requires ValidLabel(1', S')
[
var slipOfS := slipOf(S, 1);
var slipOfS' := slipOf(S', l');

match slipOfS |
case Skip \Rightarrow IsSkip(slipOfS')
case Assignment(LHS,RHS) \Rightarrow IsAssignment(slipOfS')
case IF(B, Sthen, Selse) \Rightarrow IsIF(slipOfS') \∧
  match slipOfS' |
    case IF(B', Sthen', Selse') \Rightarrow IsIF(Sthen') \∧ IsIF(Selse')
  |
case DO(B, Sloop) \Rightarrow IsSeqComp(slipOfS') \∧
  match slipOfS' |
    case SeqComp(S1', S2') \Rightarrow IsSeqComp(S1') \∧ IsDO(S2') \∧
      match S2' |
        case DO(B', Sloop') \Rightarrow IsSeqComp(Sloop')
    |
  ]
]

predicate MatchingSlipsFromSSA (S': Statement, l': Label, 
S: Statement, l: Label) 
reads *
requires Valid(S) \∧ Valid(S')
requires Core(S) \∧ Core(S')
requires ValidLabel(1, S)
requires ValidLabel(1', S')
[
  MatchingSlipsToSSA(S, l, S', l')
]

function InstanceOf(S': Statement, l': Label, v: Variable, 
XLs: seq<set<Variable>>, X: seq<Variable>,
globS: set<Variable>): Variable 
reads *
requires Valid(S') \∧ Core(S')
requires ValidXLs(globS, XLs, X)
requires ValidLabel(1', S')
[
  if l' = [] then
    assert IsAssignment(S');
    var v' : | v' \in setOf(GetLHS(S')) \* InstancesOf(S', v, X, XLs, globS);
    v'
  else
    match S' |
      case SeqComp(S1', S2') \Rightarrow
        if l'[0] = 1 then InstanceOf(S1', l'[1..], v, XLs, X, globS)
        else InstanceOf(S2', l'[1..], v, XLs, X, globS)
      | case IF(B0', Sthen', Selse') \Rightarrow
]
if \( l'[0] = 1 \) then \( \text{InstanceOf}(\text{Sthen}', l'[1..], v, \text{XLs}, X, \text{globS}) \)
\( \text{else} \) \( \text{InstanceOf}(\text{Selse}', l'[1..], v, \text{XLs}, X, \text{globS}) \)
case \( \text{DO}(B', \text{Sloop}') \) \( \Rightarrow \)
\( \text{InstanceOf}(\text{Sloop}', l'[1..], v, \text{XLs}, X, \text{globS}) \)
\}
\}

\textbf{function} \( \text{InstancesOf}(S: \text{Statement}, v: \text{Variable}, X: \text{seq<Variable>}, \text{XLs: seq<set<Variable>>}, \text{globS: set<Variable>}) : \text{set<Variable>} \)
\( \text{requires} \ \text{ValidXLs}(\text{globS}, \text{XLs}, X) \)
\[
\text{if} \ X = [] \ \text{then} \ \{ \}
\text{else if} \ X[0] = v \ \text{then} \ \text{XLs}[0]
\text{else} \ \text{InstancesOf}(S, v, X[1..], \text{XLs}[1..], \text{globS})
\]

\textbf{predicate} \( \text{ValidXLs}(\text{globS: set<Variable>}, \text{XLs: seq<set<Variable>>}, \text{X: seq<Variable>}) \)
\[
|X| = |XLs| \land
(\forall i, j \bullet 0 \leq i < j < |XLs| \implies XLs[i] \cap XLs[j]) \land
(\forall s \bullet s \text{ in } XLs \implies s \cap globS)
\]

\textbf{function} \( \text{Rename}(S': \text{Statement}, \text{XLs: seq<set<Variable>>}, \text{X: seq<Variable>}, \text{globS: set<Variable>}) : \text{Statement} \)
\( \text{reads} \ * \)
\( \text{requires} \ \text{Valid}(S') \land \text{Core}(S') \)
\( \text{requires} \ \text{ValidXLs}(\text{globS}, \text{XLs}, X) \)
\( \text{ensures} \ \text{Valid}(\text{Rename}(S', \text{XLs}, X, \text{globS})) \land \)
\( \text{Core}(\text{Rename}(S', \text{XLs}, X, \text{globS})) \)
\[
\text{match} \ S' \{
\text{case} \ \text{Assignment}(\text{LHS}, \text{RHS}) \Rightarrow
\text{RenameAssignment}(\text{LHS}, \text{RHS}, \text{XLs}, X, \text{globS})
\text{case} \ \text{SeqComp}(S1, S2) \Rightarrow
\text{SeqComp}(\text{Rename}(S1, \text{XLs}, X, \text{globS}), \text{Rename}(S2, \text{XLs}, X, \text{globS}))
\text{case} \ \text{IF}(B0, \text{Sthen}, \text{Selse}) \Rightarrow
\text{IF}(\text{RenameBoolExp}(B0, \text{XLs}, X), \text{Rename}(\text{Sthen}, \text{XLs}, X, \text{globS}), \text{Rename}(\text{Selse}, \text{XLs}, X, \text{globS}))
\text{case} \ \text{DO}(B, S1) \Rightarrow
\text{DO}(\text{RenameBoolExp}(B, \text{XLs}, X), \text{Rename}(S1, \text{XLs}, X, \text{globS}))
\text{case} \ \text{Skip} \Rightarrow \text{Skip}
\}
\]

\textbf{function} \( \text{RemoveEmptyAssignments}(S: \text{Statement}) : \text{Statement} \)
\( \text{requires} \ \text{Core}(S) \land \text{Valid}(S) \)
\[
\text{match} \ S \{
\text{case} \ \text{Assignment}(\text{LHS}, \text{RHS}) \Rightarrow
\text{if} \ |\text{LHS}| = 0 \ \text{then} \ \text{Skip} \ \text{else} \ S
\text{case} \ \text{SeqComp}(S1, S2) \Rightarrow
\text{if} \ \text{IsEmptyAssignment}(S1) \ \text{then}
\text{RemoveEmptyAssignments}(S2)
\text{else if} \ \text{IsEmptyAssignment}(S2) \ \text{then}
\text{RemoveEmptyAssignments}(S1)
\text{else}
\text{SeqComp}(\text{RemoveEmptyAssignments}(S1), \text{RemoveEmptyAssignments}(S2))
\text{case} \ \text{IF}(B0, \text{Sthen}, \text{Selse}) \Rightarrow
\text{IF}(B0, \text{RemoveEmptyAssignments}(\text{Sthen}), \text{RemoveEmptyAssignments}(\text{Selse}))
\}
\]
case DO(B,S1) ⇒
    DO(B, RemoveEmptyAssignments(S1))
case Skip ⇒ Skip
|
| predicate IsEmptyAssignment(S: Statement)
    requires Valid(S) ∧ Core(S)
    | IsAssignment(S) ∧ |GetLHS(S)| = 0
|
predicate NoEmptyAssignments(S: Statement)
    reads *
    requires Valid(S) ∧ Core(S)
    |
    match S {
    case Assignment(LHS,RHS) ⇒
        LHS ≠ [] ∧ RHS ≠ []
    case SeqComp(S1,S2) ⇒
        NoEmptyAssignments(S1) ∧ NoEmptyAssignments(S2)
    case IF(B0,Sthen,Selse) ⇒
        NoEmptyAssignments(Sthen) ∧ NoEmptyAssignments(Selse)
    case DO(B,S1) ⇒
        NoEmptyAssignments(S1)
    case Skip ⇒ true
    |
    | predicate NoSelfAssignments(S: Statement)
    reads *
    requires Valid(S) ∧ Core(S)
    |
    match S {
    case Assignment(LHS,RHS) ⇒
        NoSelfAssignmentsInAssignment(LHS, RHS)
    case SeqComp(S1,S2) ⇒
        NoSelfAssignments(S1) ∧ NoSelfAssignments(S2)
    case IF(B0,Sthen,Selse) ⇒
        NoSelfAssignments(Sthen) ∧ NoSelfAssignments(Selse)
    case DO(B,S1) ⇒
        NoSelfAssignments(S1)
    case Skip ⇒ true
    |
    | predicate NoSelfAssignmentsInAssignment(LHS: seq<Variable>,
        RHS: seq<Expression>)
    reads *
    requires Valid(Assignment(LHS, RHS))
    |
    if LHS = [] then true
    else
        if LHS[0] = GetFirstVariableInRHS(RHS[0]) then false
        else NoSelfAssignmentsInAssignment(LHS[1..], RHS[1..])
    |}

A.2 Full proofs of theorems and lemmas

function RF(S′: Statement, v′: Variable): set<VarSlide>
    reads *
requires Valid(S’) ∧ Core(S’)

if v’ ∉ def(S’) then {}
else
  var varSlides := VarSlideDGVarSlides(VarSlideDGOf(S’));
  var vSlide := VarSlideOfInstance(S’, v’);

if VarSlideTag(vSlide) = Regular then {vSlide}
else
  var instances := VarSlideInstances(vSlide, S’);
  (set i | i in instances ∧ i in def(S’) ∧ VarSlideTag(VarSlideOfInstance(S’, i)) = Regular • VarSlideOfInstance(S’, i)) +
  (set i1, i2 | i1 in instances ∧ i1 in def(S’) ∧ VarSlideTag(VarSlideOfInstance(S’, i1)) = Phi ∧ i2 in RF(S’, i1) • i2)
}

function VarSlideOfInstance(S’: Statement, v’: Variable): VarSlide
  reads *
  requires Valid(S’) ∧ Core(S’)
  requires v’ in def(S’)
  {
    var varSlideDG := VarSlideDGOf(S’);
    var varSlides := VarSlideDGVarSlides(varSlideDG);
    var vSlide := vSlide in varSlides ∧ VarSlideVariable(vSlide) = v’;
    vSlide
  }

function VarSlideInstances(vSlide: VarSlide, S’: Statement): set<Variable>
  reads *
  requires Valid(S’) ∧ Core(S’)
  requires vSlide in VarSlidesOf(S’, def(S’))
  {
    (set l’, i | i in VarSlideLabels(vSlide, S’) ∧ IsAssignment(slipOf(S’, l’)) ∧ i in UsedVarsFor(S’, l’, VarSlideVariable(vSlide)) • i)
  }

function statementSize(S: Statement): nat
  reads *
  requires Valid(S) ∧ Core(S)
  decreases S
  {
    match S {
      case Skip ⇒ 1
      case Assignment(LHS, RHS) ⇒ 1
      case SeqComp(S1, S2) ⇒ 1 + statementSize(S1) + statementSize(S2)
      case IF(B0, Sthen, Selse) ⇒ 1 + statementSize(Sthen) + statementSize(Selse)
      case DO(B, Sloop) ⇒ 1 + statementSize(Sloop)
    }
  }

A.2.1 Reaching Definitions and Liveness for exit

lemma ReachingAndLivenessForExit(S: Statement, S’: Statement, v: Variable, 
  1: Label, v’: Variable, l’: Label, XLs: seq<set<Variable>>, 
  X: seq<Variable>, V’: set<Variable>)
  requires Valid(S) ∧ Valid(S’)

requires Core(S) \land Core(S')
requires ValidLabel(1, S)
requires ValidLabel(1', S')
requires ValidXLS(glob(S), XLS, X)
requires v' in LiveOnExit(S', V', 1')
requires 1' = VarLabelOf(S, S', 1, XLS, X)
requires MatchingSlipsToSSA(S, 1, S', 1')
requires v' in InstancesOf(S', v, X, XLS, glob(S))
ensures varSlidesOfRDOUT(S, S', v, 1, XLS, X) = RF(S', v')
decreases S, distanceFromEntryForExit(S, 1)

if v \notin def(slipOf(S, 1)) {
  calc |
  RF(S', v');
  = [ assert v' in LiveOnEntryFor(S', V', 1') by |
    RLExitL1(S, S', V', 1, 1', v, v', XLS, X); ]
  ReachingAndLivenessForEntry(S, S', v, 1, v', 1', XLS, X, V');] 
  varSlidesOfRDIN(S, S', v, 1, XLS, X);
  = [ assert ReachingDefinitionsInFor(S, 1, v) = |
    ReachingDefinitionsOutFor(S, 1, v) by | RLExitL2(S, 1, v); ] |
  varSlidesOfRDOUT(S, S', v, 1, XLS, X); 
  ]

} else |
  match slipOf(S, 1) {
  case Skip \Rightarrow assert false;
  case Assignment(LHS, RHS) \Rightarrow 
    assert \{(v, 1)\} = ReachingDefinitionsOutFor(S, 1, v) by |
    RLExitL3(S, 1, v); ]
    var slide := (1, v);
    var varSlide := VarSlideOf(S, S', slide, XLS, X);
    assert \{varSlide\} = RF(S', v') by |
    RLExitL4(S, S', 1, 1', varSlide, v, v', XLS, X); ]
  case SeqComp(S1, S2) \Rightarrow 
    calc |
    RF(S', v');
    = [ assert v' in LiveOnExit(S', V', 1'+[2]) by |
      RLExitL5(S, S', V', 1, 1', v', v', XLS, X); ]
      ReachingAndLivenessForExit(S, S', v, 1+[2], |
      v', 1'+[2], XLS, X, V');] 
      varSlidesOfRDOUT(S, S', v, 1+[2], XLS, X);
    = [ RLExitL6(S, S', 1, v, XLS, X); ]
      varSlidesOfRDOUT(S, S', v, 1, XLS, X);
  case IF(B0, Sthen, Selse) \Rightarrow 
    var ThenPhiLabel := 1'+[1,2];
    var vT' : | vT' in GetCorrespondingExpression( |
      GetLHS(slipOf(S', ThenPhiLabel)),
      GetRHS(slipOf(S', ThenPhiLabel)), v').1;
    var ElsePhiLabel := 1'+[2,2];
    var vE' : | vE' in GetCorrespondingExpression( |
      GetLHS(slipOf(S', ElsePhiLabel)),
      GetRHS(slipOf(S', ElsePhiLabel)), v').1;

    calc |
    RF(S', v');
    = [ RLExitL7(S', V', 1', v', ThenPhiLabel, vT', |
      ElsePhiLabel, vE'); ]
      RF(S', vT') + RF(S', vE');
    = [ assert vT' in LiveOnExit(S', V', 1'+[1,1]) by |
      RLExitL8(S', V', 1', v', vT'); ]
ReachingAndLivenessForExit(S, S', v, 1+[1], v′, 1′+[1], XLs, X, V');
  assert varSlidesOfRDOUT(S, S', v, 1+[1], XLs, X, v) = RF(S', v'); }
varSlidesOfRDOUT(S, S', v, 1+[1], XLs, X) + RF(S', vE');
= [ assert vE' in LiveOnExit(S', V', 1'+[2,1]) by {
  RLExitL9(S', V', 1', v', vE'); }
  ReachingAndLivenessForExit(S, S', v, 1+[2], vE', 1'+[2,1], XLs, X, V');
  assert varSlidesOfRDOUT(S, S', v, 1+[2], XLs, X) = RF(S', vE'); ]
varSlidesOfRDOUT(S, S', v, 1+[1], XLs, X) +
varSlidesOfRDOUT(S, S', v, 1+[2], XLs, X);
= [ RLExitL10(S, S', 1, v, v', XLs, X); ]
varSlidesOfRDOUT(S, S', v, 1, XLs, X);
}
case DO(B, Sloop) ⇒
  var InitPhiLabel := 1'+[1];
  var vl' in GetCorrespondingExpression(
    GetLHS(slipOf(S', InitPhiLabel)),
    GetRHS(slipOf(S', InitPhiLabel)), v').1;
  var BodyPhiLabel := 1'+[2,1,2];
  var vb' in GetCorrespondingExpression(
    GetLHS(slipOf(S', BodyPhiLabel)),
    GetRHS(slipOf(S', BodyPhiLabel)), v').1;

  calc |
  RF(S', v');
  = [ RLExitL11(S', V', 1', v', InitPhiLabel, vl',
    BodyPhiLabel, vb'); ]
  RF(S', vl') + RF(S', vb');
  = [ assert vl' in LiveOnEntryFor(S', V', 1') by {
    RLExitL12(S', V', 1', v', vl'); }
    ReachingAndLivenessForEntry(S, S', v, 1, vl', 1', XLs, X, V');
    assert varSlidesOfRDIN(S, S', v, 1, XLs, X) = RF(S', vl'); ]
  varSlidesOfRDIN(S, S', v, 1, XLs, X) + RF(S', vb');
  = [ assert vb' in LiveOnExit(S', V', 1'+[2,1,1]) by {
    RLExitL13(S', V', 1', v', vb'); }
    ReachingAndLivenessForExit(S, S', v, 1+[1],
    vb', 1'+[2,1,1], XLs, X, V');
    assert varSlidesOfRDOUT(S, S', v, 1+[1], XLs, X) = RF(S', vb'); ]
  varSlidesOfRDIN(S, S', v, 1, XLs, X) +
  varSlidesOfRDOUT(S, S', v, 1+[1], XLs, X);
  = [ RLExitL14(S, S', V', 1, 1', v, v', XLs, X); ]
  varSlidesOfRDOUT(S, S', v, 1, XLs, X);
}

function varSlidesOfRDOUT(S: Statement, S': Statement, v: Variable,
  l: Label, XLs: seq<set<Variable>>, X: seq<Variable>): set<VarSlide>
reads *
requires Valid(S) ∧ Valid(S')
requires Core(S) ∧ Core(S')
requires ValidXLs(glob(S), XLs, X)
requires ValidLabel(1, S)
requires S = RemoveEmptyAssignments(Rename(S', XLs, X, glob(S)))
| (set pair | pair in ReachingDefinitionsOutFor(S, 1, v) •
  var slide := (pair.1, pair.0); VarSlideOf(S, S', slide, XLs, X))
|}
function distanceFromEntryForExit(S: Statement, l: Label): nat
reads *
requires Valid(S) ∧ Core(S) ∧ ValidLabel(1, S)
decreases S
[
  if l = [] then 2 * statementSize(S)
  else
    assert ~IsSkip(S) ∧ ~IsAssignment(S);
    match S {
    case SeqComp(S1, S2) ⇒
      if l[0] = 1 then 1 + distanceFromEntryForExit(S1, l[1..])
      else 1 + 2 * statementSize(S1) + distanceFromEntryForExit(S2, l[1..])
    case IF(B0, Sthen, Selse) ⇒
      if l[0] = 1 then 1 + distanceFromEntryForExit(Sthen, l[1..])
      else 1 + 2 * statementSize(Sthen) + distanceFromEntryForExit(Selse, l[1..])
    case DO(B, Sloop) ⇒
      1 + distanceFromEntryForExit(Sloop, l[1..])
    }
}

reads *
requires Valid(S) ∧ Valid(S’)
requires Core(S) ∧ Core(S’)
requires ValidLabel(1, S)
requires ValidLabel(1’, S’)
requires ValidXLs(glob(S), XLs, X)
requires RemoveEmptyAssignments(Rename(S’, XLs, X, glob(S))) = S
requires l’ = VarLabelOf(S, S’, 1, XLs, X)
requires v’ in InstancesOf(S’, v, X, XLs, glob(S))
requires v’ in OneLiveExit(S’, V’, l’)
[
  ∀ u’ • u’ in InstancesOf(S’, v, X, XLs, glob(S)) ∧ u’ ≠ v’ ∧ IsAssignment(slipOf(S’, 1’)) ⇒ u’ /∈ GetLHS(slipOf(S’, 1’))
]

A.2.2 Reaching Definitions and Liveness for entry

requires Valid(S) ∧ Valid(S’)
requires Core(S) ∧ Core(S’)
requires ValidLabel(1, S)
requires ValidLabel(1’, S’)
requires ValidXLs(glob(S), XLs, X)
requires RemoveEmptyAssignments(Rename(S’, XLs, X, glob(S))) = S
requires v’ in LiveOnEntryFor(S’, V’, l’)
requires l’ = VarLabelOf(S, S’, 1, XLs, X)
requires MatchingSlipsToSSA(S, 1, S’, 1’)
requires v’ in InstancesOf(S’, v, X, XLs, glob(S))
ensures varSlidesOfRDIN(S, S’, v, l, XLs, glob(S)) = RF(S’, v’)
decreases S, distanceFromEntryForEntry(S, 1)
[
  if l = [] {
    var emptySet := [];
    calc |
      RF(S’, v’);
  = { assert v’ /∈ def(S’) by { RLEntryL1(S’, V’, v’); } }
}
emptySet;
= { assert ReachingDefinitionsInFor(S, [], v) = {}; }  
varSlidesOfRDIN(S, S', v, l, XLs, X);
}
else {  
var l1, c : | l1 + [c] = 1;
m assert l1 + [c] = 1';
asser t IsSeqComp(slipOf(S', l1'));
}
else {  
asser t c = 2;
asser t + [c] = 1';
asser t IsIF(slipOf(S', l1'));
}
assert  
asser t + [c] = 1';
asser t IsSeqComp(slipOf(S', l1'));
i f (v in def(slipOf(S, 1))) {  
var Sloop' := slipOf(S', l1');  
var InitPhiLabel := l1'+[1];  
var vl' : | vl' in GetCorrespondingExpression(  
GetLHS(slipOf(S', InitPhiLabel)),
  RF(S', v');
= { assert v' in LiveOnEntryFor(S', V', l1'+[1]) by {  
REntryL6(S', V', l1'+[1], v');  
ReachingAndLivenessForEntry(S, S', v', l1', l1'+[1], v', v', l1'+[1], XLs, X, V');  
varSlidesOfRDIN(S, S', v, l1, XLs, X);  
= { REntryL7(S', l1', l1, v, XLs, X);  
varSlidesOfRDIN(S, S', v', l1, XLs, X);  
= { REntryL5(S', l1', l1, v, XLs, X);  
varSlidesOfRDIN(S', v, l1 + [2], XLs, X);  
= { RLEntryL4(S', V', l1' + [1], XLs, X, V');  
varSlidesOfRDOUT(S, S', v, l1' + [1], XLs, X);
= { RLEntryL3(S, S', l1, l1, v, XLs, X);  
varSlidesOfRDIN(S, S', v, l, XLs, X);  
= { RLEntryL2(S', V', l1', l1', v');  
ReachingAndLivenessForEntry(S, S', v, l1, v', l1', XLs, X, V');  
varSlidesOfRDIN(S, S', v, l1, XLs, X);
= { RLEntryL1(S, S', l1, l1, v, XLs, X);  
varSlidesOfRDIN(S, S', v, l, XLs, X);  
}
}
}
}
}
}
}
}
}
}
}

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GetRHS(slipOf(S', InitPhiLabel)), v').1;

var BodyPhiLabel := ll'+[2,1,2];

var vB' := [ vB' in GetCorrespondingExpression( GetLHS(slipOf(S', BodyPhiLabel)), GetRHS(slipOf(S', BodyPhiLabel)), v').1;

calc |
RF(S', v');
= { RLEntryL8(S', V', ll', v', InitPhiLabel, vI', BodyPhiLabel, vB'); } RF(S', v'I') + RF(S', vB');
= { assert vI' in LiveOnEntryFor(S', V', ll') by |
RLEntryL9(S', V', ll', 1', v', vI'); }
ReachingAndLivenessForEntry(S, S', v, ll', vI', ll', Xls, X, V');
assert varSlidesOfRDIN(S, S', v, ll, Xls, X) = RF(S', vI'); }
varSlidesOfRDIN(S, S', v, ll, Xls, X) + RF(S', vB');
= { assert vB' in LiveOnExit(S', V', ll') by |
RLEntryL10(S', V', ll', 1', v', vB'); } assert ll' + [2,1,1] = ll';
ReachingAndLivenessForExit(Sloop, Sloop', v, [], vB', [], Xls, X, {vB'});
assert varSlidesOfRDOUT(Sloop, Sloop', v, [], Xls, X) = RF(Sloop', vB');
assert RF(S', vI') + RF(Sloop', vB') = RF(S', vB'); } varSlidesOfRDIN(S, S', v, ll, Xls, X) +
varSlidesOfRDOUT(Sloop, Sloop', v, [], Xls, X);
= { RLEntryL11(S, S', Sloop, Sloop', V', ll, 1, v, Xls, X); } varSlidesOfRDIN(S, S', v, ll, Xls, X);
}
}
else |

calc |
RF(S', v');
= { RLEntryL12(S', V', ll', 1', v'); } assert v' in LiveOnEntryFor(S', V', ll') by |
RLEntryL13(S', V', ll', 1', v'); }
ReachingAndLivenessForEntry(S, S', v, ll', v', ll', Xls, X, V'); } varSlidesOfRDIN(S, S', v, ll, Xls, X);
= { RLEntryL14(S, S', 1, ll, v, Xls, X); } varSlidesOfRDIN(S, S', v, 1, Xls, X);
}
}
}
}

function varSlidesOfRDIN(S: Statement, S': Statement, v: Variable, ll: Label, Xls: seq<set<Variable>>, X: seq<Variable>): set<VarSlide> reads -
requires Valid(S) ∧ Valid(S')
requires Core(S) ∧ Core(S')
requires ValidXLS(glob(S), Xls, X)
requires ValidLabel(1, S)
requires S = RemoveEmptyAssignments(Rename(S', Xls, X, glob(S)))
{
(set pair | pair in ReachingDefinitionsInFor(S, 1, v) •
var slide := (pair.1, pair.0); VarSlideOf(S, S', slide, Xls, X))
}

function distanceFromEntryForEntry(S: Statement, ll: Label): nat reads -
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requires Valid(S) ∧ Core(S) ∧ ValidLabel(1, S)
decreases S
if 1 = [] then 1
else
  assert ¬IsSkip(S) ∧ ¬IsAssignment(S);
match S {
case SeqComp(S1, S2) ⇒
  if l[0] = 1 then 1 + distanceFromEntryForEntry(S1, l[1..])
  else 1 + 2*statementSize(S1) + distanceFromEntryForEntry(S2, l[1..])
case IF(B0, Sthen, Selse) ⇒
  if l[0] = 1 then 1 + distanceFromEntryForEntry(Sthen, l[1..])
  else 1 + 2*statementSize(Sthen) + distanceFromEntryForEntry(Selse, l[1..])
case DO(B, Sloop) ⇒
  1 + distanceFromEntryForEntry(Sloop, l[1..])
}

A.2.3 Identical Skip Slips

Lemma 11 (LemmaIdenticalSkipSlips).

requires Valid(S) ∧ Valid(SV) ∧ Valid(SV') ∧ Valid(res)
requires Core(S) ∧ Core(SV) ∧ Core(SV') ∧ Core(res)
requires SliceOf(S,V).1 = SV
requires RemoveEmptyAssignments(Rename(SV', XLS, X, glob(res))) = res
requires RemoveEmptyAssignments(Rename(S', XLS, X, glob(res))) = S
requires ∀ Sm • Sm in slidesSV ⇐⇒ (Sm in SlidesOf(S, def(S)) ∧ ∃ Sn • Sn in FinalDefSlides(S, V) ∧ SlideDGReachable(SlideDGOf(S), Sm, Sn, SlideDGSlides(SlideDGOf(S))))
requires ∀ vSlide • vSlide in varSlidesSV ⇒ (vSlide in VarSlidesOf(S', def(S)) ∧ ∃ Sn: VarSlide • VarSlideVariable(Sn) in V' ∧ VarSlideDGReachable(VarSlideDGOf(S'), vSlide, Sn, VarSlideDGVarSlides(VarSlideDGOf(S'))))
requires ∀ slide • slide in SlideDGSlides(SlideDGOf(S)) ⇒ (slide in slidesSV ⇐⇒ VarSlideOf(S, S', slide, XLS, X) in varSlidesSV)
requires ValidLabel(1, S) ∧ ValidLabel(1, SV) ∧ ValidLabel(1, res)
requires MatchingSlips(S, res, 1)
ensures IsSkip(slipOf(SV, 1)) ⇐⇒ IsSkip(slipOf(res, 1))
|
assert IsSkip(slipOf(SV, 1)) ⇐⇒ IsSkip(slipOf(res, 1)) by |
if (IsSkip(slipOf(SV, 1))) |
  LemmaIdenticalSkipSlipsA(S, S', V, SV, res, SV', V', XLS, X, 1, slidesSV, varSlidesSV); |
else |
  LemmaIdenticalSkipSlipsB(S, S', V, SV, res, SV', V', XLS, X, 1, slidesSV, varSlidesSV); |
|
lemma LemmaIdenticalSkipSlipsA(S: Statement, S': Statement,
Lemma Identical Skip Slips B

1: Label, slidesSV: set<Slide>, varSlidesSV: set<VarSlide>
requires Valid(S) ∧ Valid(SV) ∧ Valid(SV') ∧ Valid(res)
requires Core(S) ∧ Core(SV) ∧ Core(SV') ∧ Core(res)
requires SliceOf(S, V).1 = SV
requires RemoveEmptyAssignments(Rename(SV', XLs, X, glob(res))) = res
requires RemoveEmptyAssignments(Rename(S', XLs, X, glob(res))) = S
requires ∀ Sm • Sm in slidesSV ⇔ (Sm in SlidesOf(S, def(S)) ∧ Sn: VarSlide ▷ VarSlideVariable(Sm) in V' ∧
  SlidesOf(SliceDGReachable(VarslidesOf(S)', Sm, Sn, SlidesOf(SliceDGOf(S')))))
requires ∀ vSlide • vSlide in varSlidesSV ⇒
  (vSlide in SlidesOf(S, def(S)) ∧
   Sn: VarSlide ▷ VarSlideVariable(Sm) in V' ∧
   vSlide in SlidesOf(SliceDGReachable(SVslicesOf(S'), Sm, Sn)) ∧
   SlidesOf(SliceDGReachable(SVslicesOf(S'), Sm, Sn)))
requires ∀ slide • slide in SlidesOf(S, V) ⇒
  (slide in SlidesOf(SliceDGReachable(SVslicesOf(S'), Sm, Sn)))
requires ∀ slide • slide in varSlidesSV ⇒
  (slide in SlidesOf(SliceDGReachable(SVslicesOf(S'), Sm, Sn)))
requires ∀ slide • slide in SlidesOf(S'V, SV, 1) ⇒
  slide in SlidesOf(S', SV) ∧
  slide in SlidesOf(SV, 1)
requires ∀ slide • slide in varSlidesSV ⇒
  (slide in SlidesOf(SliceDGReachable(SVslicesOf(S'), Sm, Sn)))

assert IsSkip(slipOf(res, 1)) by

calc {
  assert IsSkip(slipOf(res, 1));
  ⇒ [ assert slidesOfSlipSV = [] ⇔ IsSkip(slipOf(SV, 1)); ]
  slidesOfSlipSV = [];
  ⇒ [ assert varSlidesOfSlipSV = [] ⇒ IsSkip(slipOf(SV', 1)); ]
  varSlidesOfSlipSV = [];
  ⇒ [ assert varSlidesOfSlipSV' = [] ⇒ IsSkip(slipOf(SV', 1)); ]
  IsSkip(slipOf(SV', 1'));
  ⇒ [ assert MatchingSlipsFromSSA(SV', 1', res, 1) by { LemmaMatchingSlipsFromSSA(SV', 1', res, 1, XLs, X); ]
  IsSkip(slipOf(res, 1));
}

lemma Lemma Identical Skip Slips B

1: Label, slidesSV: set<Slide>, varSlidesSV: set<VarSlide>
requires Valid(S) ∧ Valid(SV) ∧ Valid(SV') ∧ Valid(res)
requires Core(S) ∧ Core(SV) ∧ Core(SV') ∧ Core(res)
requires SliceOf(S, V).1 = SV
requires RemoveEmptyAssignments(Rename(SV', XLs, X, glob(res))) = res
requires RemoveEmptyAssignments(Rename(S', XLs, X, glob(res))) = S
requires ∀ Sm • Sm in slidesSV ⇔ (Sm in SlidesOf(S, def(S)) ∧
  Sn: VarSlide ▷ VarSlideVariable(Sm) in V' ∧
  SlidesOf(SliceDGReachable(VarslidesOf(S)', Sm, Sn)))
requires ∀ vSlide • vSlide in varSlidesSV ⇒
  (vSlide in SlidesOf(SliceDGReachable(SVslicesOf(S'), Sm, Sn)))
requires ∀ slide • slide in SlidesOf(SliceDGReachable(SVslicesOf(S'))) ⇒
  (slide in SlidesOf(SliceDGReachable(SVslicesOf(S'))))


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VarSlideOf(S, S', slide, XLs, X) in varSlidesSV

requires ValidLabel(1, S) \land ValidLabel(1, SV) \land ValidLabel(1, res)
requires MatchingSlips(S, res, l)
requires \neg IsSkip(slipOf(SV, 1))
ensures \neg IsSkip(slipOf(res, 1))

\[
\begin{aligned}
&\textit{VarLabelOf}(S, SV', 1, XLs, X) \in \textit{VarSlidesSV} \\
&\textit{VarLabelOf}(S, SV', 1, XLs, X) \in \textit{VarSlidesSV} \\
&\textit{VarLabelOf}(S, SV', 1, XLs, X) \in \textit{VarSlidesSV} \\
\end{aligned}
\]

\[\{\text{assert } \neg \text{IsSkip(slipOf(res, 1)) by }\]
\[\begin{aligned}
&\text{calc } \\
&\quad \neg \text{IsSkip(slipOf(SV, 1))}; \\
&\quad \Rightarrow \ [\text{assert } \text{slidesOfSlipSV} = [] \iff \text{IsSkip(slipOf(SV, 1))}; ] \\
&\quad \Rightarrow \ [\text{assert } \text{slidesOfSlipSV} = [] \iff \text{varSlidesOfSlipSV'} = []]; \\
&\quad \Rightarrow \ [\text{assert } \text{varSlidesOfSlipSV'} = [] \iff \text{IsSkip(slipOf(SV', 1))}; ] \\
&\quad \neg \text{IsSkip(slipOf(SV', 1'))}; \\
&\quad \Rightarrow \ [\text{LemmaRenameSkip(slipOf(SV', 1'), XLS X, glob(slipOf(res, 1))}); ] \\
&\quad \neg \text{IsSkip(Rename(slipOf(SV', 1'), XLS X, glob(slipOf(res, 1)))); ] \\
&\quad \Rightarrow \ [\text{LemmaRemoveEmptyAssignmentsSkip(Rename(slipOf(SV', 1'), XLS X, glob(slipOf(res, 1)))); ] \\
&\quad \neg \text{IsSkip(slipOf(RemoveEmptyAssignments(Rename(slipOf(SV', 1'), XLS X, glob(res))); 1))); ] \\
&\quad \Rightarrow \ [\text{assert } \text{RemoveEmptyAssignments(Rename(SV', XLS X, glob(res))); res}]; \\
&\quad \neg \text{IsSkip(slipOf(res, 1))}; \\
\}\]

A.2.4 Inverse varLabelOf

Lemma 12 (LemmaInverseVarLabelOf).

function LabelOf(S': Statement, S: Statement, l': Label, XLs: seq<set<Variable>>, X: seq<Variable>): Label
reads *
requires Valid(S) \land Valid(S')
requires Core(S) \land Core(S')
requires ValidLabel(1', S')
requires ValidXLs(glob(S), XLs, X)
requires S = RemoveEmptyAssignments(Rename(S', XLs, X, glob(S)))
requires MatchingSlipsFromSSA(S', [], S, [])
ensures ValidLabel(LabelOf(S', S, 1', XLs, X, S))

\[
\begin{aligned}
&\text{match } S' \{ \\
&\qquad \text{case } \text{Skip } \Rightarrow \\
&\qquad \quad \text{assert } \text{IsSkip(S)}; \\
&\qquad \quad 1' = []; \\
&\qquad \} \\
&\qquad \text{case } \text{Assignment}([LHS', RHS']) \Rightarrow \\
&\qquad \quad \text{assert } \text{IsAssignment(S) } \lor \text{IsSkip(S)}; \\
&\qquad \quad 1' = []; \\
&\qquad \} \\
&\qquad \text{case } \text{SeqComp}(S1', S2') \Rightarrow \\
\end{aligned}
\]
if \( l' = [1] \) then \( [] \)

else

assert IsSeqComp(S) ∨ IsDO(S);

if IsSeqComp(S) then

match S |

case SeqComp(S1, S2) ⇒

if \( l'[0] = 1 \) then \([1]\) + LabelOf(S1', S1, \( l'[1..] \), XLs, X)

else \([2]\) + LabelOf(S2', S2, \( l'[1..] \), XLs, X)

} else

assert IsDO(S2') ∧ IsSeqComp(GetLoopBody(S2'));

var Sloop' := GetS1(GetLoopBody(S2'));

match S |

case DO(B, Sloop) ⇒ \([1]\) + LabelOf(Sloop', Sloop, \( l'[3..] \), XLs, X)

} else

assert IsDO(S2') ∧ IsSeqComp(GetLoopBody(S2'));

match S |

case IF(B0, Sthen, Selse) ⇒

assert IsIF(S);

if \( l' = [1] \) then \( [] \) else

match S |

case IF(B0, Sthen, Selse) ⇒ \([1]\) + LabelOf(Sthen', Sthen, \( l'[2..] \), XLs, X)

} else \([2]\) + LabelOf(Selse', Selse, \( l'[2..] \), XLs, X)

]}

lemma LemmaInverseVarLabelOfRec(S: Statement, S': Statement, SV': Statement, res: Statement, XLs: seq<set<Variable>>, X: seq<Variable>)

requires Valid(S) ∧ Valid(S') ∧ Valid(SV') ∧ Valid(res)

requires Core(S) ∧ Core(S') ∧ Core(SV') ∧ Core(res)

requires ValidXLs(glob(S), XLs, X)

requires ValidXLs(glob(res), XLs, X)

requires S = RemoveEmptyAssignments(Rename(S', XLs, X, glob(S)))

requires MatchingSlipsToSSA(S, [1], S', [1])

requires res = RemoveEmptyAssignments(Rename(SV', XLs, X, glob(res)))

requires MatchingSlipsFromSSA(SV', [1], res, [1])

\[ ∀ 1, l' \mid ValidLabel(1, S) ∧ l' = VarLabelOf(S, S', 1, XLs, X) ∧ ValidLabel(l', S') ∧ ValidLabel(l', SV') ∧ NoSelfAssignments(S) ∧ NoEmptyAssignments(S) \Rightarrow l = LabelOf(SV', res, 1', XLs, X) \]

lemma LemmaInverseVarLabelOfRec(S: Statement, S': Statement, SV': Statement, res: Statement, XLs: seq<set<Variable>>, X: seq<Variable>)

requires Valid(S) ∧ Valid(S') ∧ Valid(SV') ∧ Valid(res)

requires Core(S) ∧ Core(S') ∧ Core(SV') ∧ Core(res)

requires ValidXLs(glob(S), XLs, X)

requires ValidXLs(glob(res), XLs, X)

requires S = RemoveEmptyAssignments(Rename(S', XLs, X, glob(S)))

requires MatchingSlipsToSSA(S, [1], S', [1])

requires res = RemoveEmptyAssignments(Rename(SV', XLs, X, glob(res)))

requires MatchingSlipsFromSSA(SV', [1], res, [1])

ensures InverseVarLabelOf(S, S', SV', res, XLs, X)

\[ ∀ 1, l' \mid ValidLabel(1, S) ∧ l' = VarLabelOf(S, S', 1, XLs, X) ∧ ValidLabel(l', S') ∧ ValidLabel(l', SV') ∧ NoSelfAssignments(S) ∧ NoEmptyAssignments(S) \textbf{ensures} l = LabelOf(SV', res, 1', XLs, X) \]

lemma LemmaInverseVarLabelOfRec(S: Statement, S': Statement, SV': Statement, res: Statement, XLs: seq<set<Variable>>, X: seq<Variable>)

requires Valid(S) ∧ Valid(S') ∧ Valid(SV') ∧ Valid(res)

requires Core(S) ∧ Core(S') ∧ Core(SV') ∧ Core(res)

requires ValidXLs(glob(S), XLs, X)

requires ValidXLs(glob(res), XLs, X)

requires S = RemoveEmptyAssignments(Rename(S', XLs, X, glob(S)))

requires MatchingSlipsToSSA(S, [1], S', [1])

requires res = RemoveEmptyAssignments(Rename(SV', XLs, X, glob(res)))

requires MatchingSlipsFromSSA(SV', [1], res, [1])

ensures InverseVarLabelOf(S, S', SV', res, XLs, X)

\[ ∀ 1, l' \mid ValidLabel(1, S) ∧ l' = VarLabelOf(S, S', 1, XLs, X) ∧ ValidLabel(l', S') ∧ ValidLabel(l', SV') ∧ NoSelfAssignments(S) ∧ NoEmptyAssignments(S) \textbf{ensures} l = LabelOf(SV', res, 1', XLs, X) \]

lemma LemmaInverseVarLabelOfRec(S: Statement, S': Statement,
SV': Statement, res: Statement, l: Label, l': Label,
XLS: seq<set<Variable>>, X: seq<Variable>)
requires Valid(S) ∧ Valid(S') ∧ Valid(SV') ∧ Valid(res)
requires Core(S) ∧ Core(S') ∧ Core(SV') ∧ Core(res)
requires ValidXLS(glob(S), XLS, X)
requires ValidXLS(glob(res), XLS, X)
requires S = RemoveEmptyAssignments(Rename(S', XLS, X, glob(S)))
requires MatchingSlipsToSSA(S, [], S', [])
requires res = RemoveEmptyAssignments(Rename(SV', XLS, X, glob(res)))
requires MatchingSlipsFromSSA(SV', [], res, [])
requires ValidLabel(l, S)
requires ValidLabel(l', S')
requires ValidLabel(l', SV')
requires l' = VarLabelOf(S, S', l, XLS, X)
requires SV' = LabelOf(SV', res, l', XLS, X)
[
match SV' {
    case Skip ⇒
        assert IsSkip(res);
        assert l' = [];
        assert l = [];
    case Assignment(LHS', RHS') ⇒
        assert IsAssignment(res) ∨ IsSkip(res);
        assert l' = [];
        assert l = [];
    case SeqComp(S1', S2') ⇒
        if l' = [] { assert l = []; } else {
            assert IsSeqComp(res) ∨ IsDO(res);
            if IsSeqComp(res) {
                var S1, S2 := GetS1(res), GetS2(res);
                if l'[0] = 1 {
                    calc {
                        LabelOf(SV', res, l', XLS, X);
                        = { assert IsSeqComp(SV') ∧ IsSeqComp(res) ∧ l'[0] = 1; } 
                        [1] + LabelOf(S1', S1, l'[1..], XLS, X);
                        = { LemmaInverseVarLabelOfRec(GetS1(S), GetS1(S'), S1', S1, 
                            l'[1..], l'[1..], XLS, X); 
                            assert l'[1..] = LabelOf(S1', S1, l'[1..], XLS, X); } 
                        [1] + l'[1..];
                        = { assert l[0] = 1 by {
                            assert l' = VarLabelOf(S, S', 1, XLS, X); } } 
                        1;
                    }
                }
                else {
                    calc {
                        LabelOf(SV', res, l', XLS, X);
                        = { assert IsSeqComp(SV') ∧ IsSeqComp(res) ∧ l'[0] = 2; } 
                        [2] + LabelOf(S2', S2, l'[1..], XLS, X);
                        = { LemmaInverseVarLabelOfRec(GetS2(S), GetS2(S'), S2', S2, 
                            l'[1..], l'[1..], XLS, X); 
                            assert l'[1..] = LabelOf(S2', S2, l'[1..], XLS, X); } 
                        [2] + l'[1..];
                        = { assert l[0] = 2 by {
                            assert l' = VarLabelOf(S, S', 1, XLS, X); } } 
                        1;
                    }
                }
            }
            else {
                var Sloop' := GetS1(GetLoopBody(S2'));
                var B, Sloop := GetLoopBool(res), GetLoopBody(res);
                calc {
                    LabelOf(SV', res, l', XLS, X);
                    = { assert IsSeqComp(SV') ∧ IsDO(res); }
                }
            }
        }
}
\[ \text{LemmaInverseVarLabelOfRec(GetLoopBody(S), GetS1(GetLoopBody(GetS2(S'))), Sloop', Sloop, 1[1..], 1'[3..], XLs, X);} \]
\[ \text{assert 1[1..] = LabelOf(Sloop', Sloop, 1'[3..], XLs, X);} \]
\[ \text{assert 1[0] = 1; } \]
\[ \text{assert 1 = [1]; } \]
\[ \text{assert 1' = VarLabelOf(S, S', 1, XLs, X); } \]
\[ \text{assert 1' = VarLabelOf(S, S', 1, XLs, X); } \]
\[ \text{assert 1' = VarLabelOf(S, S', 1, XLs, X); } \]
\[ \text{assert 1' = VarLabelOf(S, S', 1, XLs, X); } \]
Bibliography


